Math 1580: Cryptography *Lecture Notes*

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§2.1 Linear Combinations *continued*

Recall from last time that we proposed that

$$\text{greatest common divisor} \leq \text{least linear combination}.$$

Example 2.1

$\gcd(2024, 748) = 44$ because we have

$$\begin{aligned} 2024 &= 748 \cdot 2 + 528 \\ 748 &= 528 \cdot 1 + 220 \\ 528 &= 220 \cdot 2 + 88 \\ 220 &= 88 \cdot 2 + \boxed{44} \leftarrow \gcd(2024, 748) \\ 88 &= 44 \cdot 2 + 0 \end{aligned}$$

We determine which linear combinations or 2024 and 748 we can create:

$$\begin{aligned}
2024 &= 1 \cdot 2024 + 0 \cdot 748 \\
748 &= 0 \cdot 2024 + 1 \cdot 748 \\
528 &= 1 \cdot 2024 + (-2) \cdot 748 \\
220 &= 748 - 1 \cdot 528 \\
&= 748 - 1 \cdot (1 \cdot 2024 + (-2) \cdot 748) \\
&= -1 \cdot 2024 + 3 \cdot 748 \\
88 &= 528 - 2 \cdot 220 \\
&= \underbrace{[1 \cdot 2024 + (-2) \cdot 748]}_{528} - 2 \cdot \underbrace{[-1 \cdot 2024 + 3 \cdot 748]}_{220} \\
&= 3 \cdot 2024 - 8 \cdot 748 \\
44 &= 220 - 2 \cdot 88 \\
&= [-1 \cdot 2024 + 3 \cdot 748] - 2 \cdot [3 \cdot 2024 - 8 \cdot 748] \\
&= -7 \cdot 2024 + 19 \cdot 748
\end{aligned}$$

Following this example, we have shown that every common divisor of a and b can be written as a linear combination of a and b, and since the greatest common divisor has to be less than the least linear combination (as shown last time), the greatest common divisor *is* the least linear combination[1].

We realize that there is a *recurrence* happening here. If we call every set of coefficients x, y and z, w for a and b respectively, such that

$$\begin{aligned}
a &= x \cdot a_0 + y \cdot b_0 \\
b &= z \cdot a_0 + y \cdot b_0
\end{aligned}$$

where a_0 and b_0 are the original numbers, we can use a sliding window approach[2] again to determine the next set of x, y, z, w, a, b.

Recall from last time we had

$$\begin{aligned}
a' &= b \\
b' &= a \mod b
\end{aligned}$$

We can extend this algorithm for our new coefficients:

$$\begin{aligned}
x' &= z \\
y' &= w \\
z' &= w - \left\lfloor \frac{a}{b} \right\rfloor \cdot z \\
w' &= y - \left\lfloor \frac{a}{b} \right\rfloor \cdot w
\end{aligned}$$

[1] Assume for contradiction that the gcd were any less, then that would also be a linear combination. $\lightning$

[2] Updating our iterators on every loop by sliding our window of coefficients down.

where $\left\lfloor \frac{a}{b} \right\rfloor$ are the quotients from our Euclidean Algorithm. Note that initially, we have

$$a = 1 \cdot a_0 + 0 \cdot b_0$$
$$b = 0 \cdot a_0 + 1 \cdot b_0$$

so we have initial values of $x = 1, y = 0, z = 0, w = 0$.

so our code for the *extended Euclidean Algorithm* is now

Algorithm 2.2 (Extended Euclidean Algorithm) —

```
def ext_gcd(a: int, b: int) -> tuple[int, int]:
    """
    Computes the gcd of a and b using the extended Euclidean algorithm.
    param a: int
    param b: int
    return: tuple (int x, int y) where ax + by = gcd(a, b)
    """
    x, y, z, w = 1, 0, 0, 1
    while b != 0:
        x, y, z, w = z, w, x - (a // b) * z, y - (a // b) * w
        a, b = b, a % b
    return (x, y)
```

§2.2 Modular Arithmetic

Recall: We used a substitution/shift cipher to encrypt text:

Y	E	S
↓	↓	↓
D	J	X

by incrementing 5 letters for each lecture.

$a = 0$, $b = 1$, $\ldots$, $z = 25$.

We had this notion of

$$\text{ciphertext} = \text{plaintext} + 5$$
$$\mathtt{d} = \mathtt{y} + 5$$
$$3 = 24 + 5 = 29$$

Definition 2.3

We say $a \equiv b \mod m$ if $m \mid a - b$.

We say "a is congruent[a] to b modulo m".

[a] Congruence is a "behave like" equality.

Example 2.4

$$24 + 5 \equiv 3 \mod 26$$
$$22 + 2 \equiv 1 \mod 12$$

The first example is from our shift sipher, the second example is equivalent to "two hours after 11:00, it is 1:00".

Proposition 2.5

If we have

$$a_1 \equiv a_2 \mod m$$
$$b_1 \equiv b_2 \mod m$$

Then we have the following:

$$a_1 + b_1 \equiv a_2 + b_2 \mod m \tag{1}$$
$$a_1 - b_1 \equiv a_2 - b_2 \mod m \tag{2}$$
$$a_1 \cdot b_1 \equiv a_2 \cdot b_2 \mod m \tag{3}$$

Proof. For eq. (1), realize that we have

$$(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2)$$

and the two terms on the right are each divisible by m by our premise. We can also write out

$$\begin{aligned} a_1 + b_1 &= (a_2 + \alpha m) + (b_2 + \beta m) \\ &= (a_2 + b_2) + (\alpha + \beta) \cdot m. \end{aligned}$$

Similarly, for eq. (2), we have

$$\begin{aligned} a_1 - b_1 &= a_2 + \alpha m - (b_2 + \beta m) \\ &= a_2 - b_2 + (\alpha - \beta) \cdot m. \end{aligned}$$

and for eq. (3), we have

$$\begin{aligned} a_1 \cdot b_1 &= (a_2 + \alpha m) \cdot (b_2 + \beta m) \\ &= a_2 \cdot b_2 + \alpha m b_2 + \beta m a_2 + \alpha\beta m^2 \\ &= a_2 \cdot b_2 + (\alpha b_2 + \beta a_2 + \alpha\beta m) \cdot m. \end{aligned}$$

which concludes the proofs of the premod rules. □

Proposition 2.6

There exists b with

$$a \cdot b \equiv 1 \mod m$$

if and only if $\gcd(a, m) = 1$.

Proof. We can write linear combination equation

$$a \cdot b + m \cdot k = 1$$

and we have that the following are equivalent (we cascade down the list and can easily prove the iff relations):

i. such a b exists,

ii. there is a solution b, k to this equation,

iii. 1 is a linear combination of a and m,

iv. 1 is the *least* linear combination of a and m,

v. $1 = \gcd(a, m)$.

so we have that $1 = \gcd(a, m)$ if and only if a's inverse b exists. □

§3 February 2, 2022

§3.1 Inverses mod m

Recall: Last time, we showed in proposition 2.6 that there exists an integer b with with $a \cdot b \equiv 1 \mod m$ iff $\gcd(a, m) = 1$.

Claim 3.1 — We further claim that if such a b exists, then it is unique mod m.

That is, if we have

$$\begin{aligned} a \cdot b_1 &\equiv 1 \pmod{m} \\ a \cdot b_2 &\equiv 1 \pmod{m} \end{aligned}$$

then we have that $b_1 \equiv b_2 \pmod{m}$.

Proof. We consider $b_1 a b_2$. We have

$$b_2 \equiv (b_1 a) b_2 = b_2 (a b_2) \equiv b_2$$

all taking mod m. □

How, then, could we compute this inverse b efficiently?

Recall that last class, we used the extended Euclidean algorithm to compute the linear combination of a and m efficiently,

$$\begin{aligned} 1 &= a \cdot u + m \cdot v \\ &\equiv a \cdot \boxed{u} \mod m \end{aligned}$$

where u is b.

§3.2 Modular Arithmetic *continued*

Definition 3.2 (Ring of Integers mod m)
$\mathbb{Z}/m\mathbb{Z} = \{0, 1, 2, \ldots, m-1\}$ with operations $+, -, \times \pmod{m}$.

Example 3.3
$\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$. We have the following operation tables for $\mathbb{Z}/4\mathbb{Z}$:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Definition 3.4 (Group of Units mod m)
We have the set of units in $\mathbb{Z}/m\mathbb{Z}$ as

$$\begin{aligned} (\mathbb{Z}/m\mathbb{Z})^\times &= \{a \in \mathbb{Z}/m\mathbb{Z} \mid \exists b \text{s.t. } a \cdot b \equiv 1\} \\ &= \{a \in \mathbb{Z}/m\mathbb{Z} \mid \gcd(a, m) = 1\} \end{aligned}$$

Example 3.5
$(\mathbb{Z}/4\mathbb{Z})^\times = \{1, 3\}$.

Definition 3.6 (Euler Totient Function)
We have
$$\varphi(m) = \#(\mathbb{Z}/m\mathbb{Z})^\times$$
which counts the number of units modulo m.

Example 3.7
$\varphi(4) = 2$.

Let's investigate the properties of units. Let's say a_1, a_2 are units. Which of the following have to be units?

	Does this have to be a unit?
$a_1 \cdot a_2$	Yes! Since $\gcd(a_1, m) = 1$ and $\gcd(a_2, m) = 2$ so we have $\gcd(a_1a_2, m) = 1$. We also have $a_1b_1 \equiv 1 \mod m$ and $a_2b_2 \equiv 1 \mod m$, we have $(a_1a_2)(b_2b_1) \equiv 1 \mod m$.
$a_1 + a_2$	No. We have counterexample $m = 4$: $1 + 1$ is not a unit.
$a_1 - a_2$	Also no. For any a, $a - a = 0$ which is never a unit.

Definition 3.8 (Prime Number)
An integer $n \geq 2$ is prime if its only (positive) divisors are 1 and n.

Example 3.9
Numbers like $2, 3, 5, 7, 11, 12, \ldots$.

What if m is a prime number? Then we have
$$(\mathbb{Z}/m\mathbb{Z})^\times = \{1, 2, \ldots, m-1\}$$
so we can divide by elements of $\mathbb{Z}/m\mathbb{Z}$, just like in $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. We can divide by any nonzero element of $\mathbb{Z}/m\mathbb{Z}$. We call these fields!

§3.3 Fast*ish* Powering

Problem. How might we compute $g^a \mod m$?

A naïve solution might be

```
def pow_mod(g, a, m):
    return g ** a % m
```

What if we tried to compute pow_mod(239418762304, 12349876234, 12394876123482783641) or something of the like? Something like this...

We could do something a bit more clever, like taking a mod every time we multiply:

```
def pow_mod(g, a, m):
    p = 1
    for i in range(a):
        p = (p * q) % m
    return p
```

Yet we *still* couldn't do pow_mod(239418762304, 12349876234, 12394876123482783641) since that takes the amount of time proportional to a[3].

Example 3.10

[3] Which can become big...

Let's try to compute 3^{37} by hand.

$$\begin{aligned}
3^1 &\equiv 3 \mod 100\\
3^2 &\equiv 9 \mod 100\\
3^4 = (3^2)^2 = &\equiv 81 \mod 100\\
3^8 = (3^4)^2 = 81^2 = 6561 &\equiv 61 \mod 100\\
3^{16} = (3^8)^2 \equiv 61^2 = 3721 &\equiv 21 \mod 100\\
3^{32} = (3^{16})^2 \equiv 21^2 = 441 &\equiv 41 \mod 100
\end{aligned}$$

Since $37 = 32 + 4 + 1$, we can simply do

$$3^{37} = 3^{32} \cdot 3^4 \cdot 3^1 = 41 \cdot 81 \cdot 3 = 1863 \equiv 63 \mod 100$$

§4 February 4, 2022

§4.1 Fast Powering *continued*

Example 4.1

Recall: we wanted to compute $3^{37} \mod 100$

$$\begin{aligned}
3^1 &\equiv 3 \pmod{100}\\
3^2 &\equiv 9\\
3^4 &\equiv 81\\
3^8 &\equiv 61\\
3^{16} &\equiv 21\\
3^{32} &\equiv 41
\end{aligned}$$

so we have

$$37 = 1 + 4 + 32 \qquad 3^{37} = 3^1 \cdot 3^4 \cdot 3^{32} \equiv 3 \cdot 81 \cdot 41 \equiv 63$$

How might we do this as an algorithm? We want to keep track of a few things, such as g (the current power), p (the multiple we are building), a (the remaining powers). This is akin to *deconstructing the power in binary and composing our product.*

Algorithm 4.2 (Fast Powering Algorithm) —

```
def pow_mod(a: int, b: int, p: int) -> int:
    """
    Computes a^b mod p using repeated squaring.
    param a: int
    param b: int
    param p: int
```

```
    return: int a^b mod p
    """
    result = 1
    while b > 0:
        if b & 1:
            result = (result * a) % p
        a = (a * a) % p
        b >>= 1
    return result
```

Example 4.3

$37 = 100101_2$, so we peel off last digits and multiply g into p.

Thinking about iterations, we have

g	p	a	a_2
3	1	37	$10010\underline{1}$
9	3	18	$10010\underline{0}$
81	3	9	$100\underline{1}$
61	43	4	$10\underline{0}$
21	43	2	$1\underline{0}$
41	43	1	$\underline{1}$
	$\boxed{63}$	0	$\underline{0}$

This algorithm takes approximately $\log_2(a)$ time to run, since it does as many steps for each digit in the binary representation of a.

§4.2 Fun Integers

Recall: An integer p is prime if $p \geq 2$ and

$$a \mid p \Rightarrow a = \pm 1, \pm p$$

Proposition 4.4

Let p be prime. Then $p \mid ab \Rightarrow p \mid a$ or $p \mid b$.

Example 4.5

p is not prime, this doesn't work. $p = 6$. $p \mid 4 \cdot 9 = 36$ but $6 \nmid 4$ and $6 \nmid 9$.

Proof. Let $g = \gcd(p, a)$. g is either 1 or p.

If $g = p$, then we have that $p = g \mid a$.

If $p = 1$, we can write this as

$$\begin{aligned} 1 = g &= p \cdot u + a \cdot v \\ b &= p \cdot ub + ab \cdot v \end{aligned}$$

since p is a multiple of p and ab is a multiple of p, we have that $p \mid b$. □

Theorem 4.6 (Fundamental Theorem of Arithmetic)
Any integer $a \geq 1$ can be factored into product of primes

$$a = p_1^{e_1} \cdots p_n^{e_n}$$

and this product of primes is *unique* up to rearrangement.[a]

[a] This is to say, $\mathbb{Z}$ is a UFD!

Example 4.7
Instead of thinking about integers, we think about $\mathbb{Z}[\sqrt{-5}]$, like

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$$

Consider

$$6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 2 \cdot 3$$

and each of $(1 + \sqrt{-5})$, $(1 - \sqrt{-5})$, 2, 3 have no divisors besides themselves and ± 1 (units).

Proof. We begin by working out an example:

Example 4.8
Let's factor 60, we can write this as

$$60 = 6 \cdot 10 = (2 \cdot 3) \cdot (2 \cdot 5) = 2^2 \cdot 3 \cdot 5.$$

What if we had different answers

$$p_1 p_2 \cdots p_t = a = q_1 q_2 \cdots q_s$$

We have that

$$\begin{aligned} p_1 \mid p_1 \cdots p_t &= q_1 \cdots q_s \\ &= q_1 (q_2 \cdots q_s) \end{aligned}$$

So we have that $p_1 \mid q_1$ or $p_1 \mid q_2 \cdots q_s$, and we go on. So p_1 has to divide *one* of q_i. But both are primes, so they are equal $p_1 = q_i$. We rearrange so q_i is q_1. We strip off p_1 and q_1 and we have

$$p_2 \cdots p_t = q_2 \cdots q_s$$

we continue until we have no factors left[4] □

Definition 4.9 (Order)

We define the order

$$\text{ord}_p(a) = \text{the power of } p \text{ in the factorization of } a$$

such that we have

$$a = \prod_p p^{\text{ord}_p(a)}$$

(This makes sense since $\text{ord}_p(a)$ *is finite for finitely many* p*.)*

Theorem 4.10 (Fermat's Little Theorem)

Let p be prime, $a \in \mathbb{Z}/p\mathbb{Z}$,

$$a^{p-1} \equiv \begin{cases} 0 & \text{if } a \equiv 0 \\ 1 & \text{otherwise} \end{cases}$$

In abstract algebra, this directly follows from Lagrange's Theorem for $\mathbb{Z}/p\mathbb{Z}$, we give another argument.

Proof. If $a \equiv 0$, this is sufficiently clear.

Let $a \not\equiv 0$. We look at the numbers

$$a, 2a, 3a, \ldots, (p-1)a$$

We consider 2 questions:

i. Are any of these divisible by p?

   No! $p \nmid a$ and $p \nmid i$ so $p \nmid ia$ for $1 \leq i < p$.

ii. Are any of these equal? i.e. $ia \equiv ja \mod p$.

   No again! a has an inverse mod p.

[4] We could also have taken a well-ordering approach to this statement, taking a to be the least such non-uniquely factorizable number and showing that by peeling off p_1 and q_1, we get a smaller such a, which is a contradiction.

So we have that this list is a permutation of $\{1, 2, \ldots, p-1\}$, that is,

$$\{1, 2, \ldots, p-1\} = \{a, 2a, \ldots, (p-1)a\} \mod p$$

we multiply these sets together[5],

$$\begin{aligned} 1 \cdot 2 \cdot 3 \cdots (p-1) &\equiv a \cdot 2a \cdots (p-1)a \mod p \\ &\equiv (1 \cdot 2 \cdots p-1)a^{p-1} 1 \cdot 2 \cdot 3 \cdot (p-1)(a^{p-1}-1) \equiv 0 \mod p \\ \implies a^{p-1} &\equiv 1 \mod p. \end{aligned}$$

Which is as desired. □

§5 February 7, 2022

§5.1 Orders mod p

Recall: If $a \not\equiv 0 \pmod p$, then we have $a^{p-1} \equiv 1 \pmod p$, which was theorem 4.10, Fermat's Little Theorem.

Definition 5.1 (Order of a mod p)

The order of $a \pmod p$ is the smallest positive k such that

$$a^k \equiv 1 \pmod p$$

This is not to be confused with definition 4.9 which is the power of p in the prime factorization of a. This is the order of a in the multiplicative group $\mathbb{Z}/p\mathbb{Z}$.

Proposition 5.2

let $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ be of order k. If $a^n \equiv 1 \pmod p$, then $k \mid n$.

In particular, $k \mid p-1$ by theorem 4.10, Fermat's Little Theorem.

Proof. We write $n = k \cdot q + r$ such that $0 \leq r < k$ ($\mathbb{Z}$ is a Euclidean domain)

$$1 \equiv a^n \equiv a^{kq+r} \equiv (a^k)^q \cdot a^r \equiv a^r$$

Since k is the minimal positive number such that $a^k \equiv 1$, then this forces $r = 0$. Then $k \mid n$. □

[5] This is truly a pro-gamer move

Theorem 5.3 (Primitive Root Theorem)
Let p be prime. Then there is a g such that

$$(\mathbb{Z}/p\mathbb{Z})^\times = \{1, g, g^2, \ldots, g^{p-2}\}.$$

We call g a primitive root or generator.

Example 5.4
$p = 5$, $(\mathbb{Z}/5\mathbb{Z})^\times = \{1, 2, 3, 4\}$.

1? No: $\{1, 1^2, 1^3\} = \{1\}$

2? Yes: $\{1, 2, 2^2, 2^3\} = \{1, 2, 4, 3\}$

3? Yes: $\{1, 3, 3^2, 3^3\} = \{1, 3, 4, 2\}$

4? No: $\{1, 4, 4^2, 4^3\} = \{1, 4\}$

Remark 5.5. In general, the number of primitive roots is $\varphi(p-1)$. (Take the group of exponents and solve for power).

§5.2 Discrete Logarithm Problem

We go on to discuss a fundamental property about exponentiation mod p. Let's fix some p and primitive root g.

Given some a, we can compute g^a efficiently

$$a \longrightarrow g^a \qquad \text{This is easy}$$
$$a \overset{?}{\longleftarrow} g^a \qquad \text{This is hard}$$

Note that

$$g^a \equiv g^b \Leftrightarrow g^{a-b} \equiv 1 \Leftrightarrow p-1 \mid a-b$$

so a is determined mod $p-1$.

Definition 5.6 (Discrete Logarithm)
The discrete logarithm of g^a is a.

This is known as the "Discrete Logarithm Problem" (DLP), which is concerned with how we can compute discrete logarithms.

This idea is fundamental to computer security! The real-world analogue is if you go to the bank after hours and deposit a check or cash into the deposit slot. It is relatively easy for one to deposit an item but hard for someone who doesn't work at the bank[6] to access that item.

§5.3 Cryptographic Systems

§5.3.1 Symmetric Cryptography

We have 3 people, *Alice*, *Bob*, and *Eve*.

Bob has a message m which he wants to send to Alice. However, everything he sends to Alice can (and is) intercepted by Eve. He wants to encrypt this message m he sends to Alice.

We say that a message $m \in \mathcal{M}$ in the space of possible messages. We have secret key $k \in \mathcal{K}$ that can encrypt m into ciphertext $c \in \mathcal{C}$ in the space of ciphertexts.

$$\left\{ \begin{array}{l} \text{Message } m \in \mathcal{M} \\ \text{Secret key } k \in \mathcal{K} \end{array} \right\} \rightsquigarrow \text{Ciphertext } c \in \mathcal{C} \longrightarrow \text{Alice} \rightsquigarrow m$$

If we fix k, we have

$$e_k(m) = e(k, m)$$
$$d_k(c) = d(k, c)$$

be our encryption and decryption functions. We usually take m to be a number, and we can encode letters to numbers (0-255) using ASCII.

In Python, this is implemented using functions like ord (character to encoding) and chr (encoding to character).

We'll just talk about transmitting numbers since we can convert freely between them and text.

Q: What do we want out of our cryptosystem?

0. The system is secure even if Eve knows the design. (Assume Eve knows the encryption and decryption functions, but so long as she doesn't know the key).

1. e, the encryption function, is easy to compute.

2. d, the decryption function, is similarly easy to compute.

3. Given $c_1, c_2, \ldots$ a collection of ciphertexts, encrypted with the *same* key k, it's hard to compute any message m_i.

[6] Say, possessing a *key* or *password*.

4. Given $(m_1, c_1), \ldots (m_n, c_n)$ some collection of messages and their encryptions, it remains difficult to compute $d_k(c)$ for $c \notin \{c_1, \ldots, c_n\}$. This is called a *"chosen plaintext attack"*.

§6 February 9, 2022

§6.1 Asymmetric/Public Key Cryptography

The premise is that we have *Alice* and *Bob* who are communicating, and *Eve* intercepts all communications between them. There is **no** communication between Alice and Bob ahead of time. A priori, it's not entirely obvious that this is possible...

We'll see that this is indeed possible!

Example 6.1

Analogy: Alice and Bob are communicating by writing messages on pieces of paper.

Symmetric cryptography is having a shared safe, Alice and Bob both have the key/know the combination to, and both can leave messages and retrieve messages.

1. Alice sets up a box with a thin slot with a lock on it. Alice has the key to this lock.

2. Bob is able to deposit messages into the slot in the box, and Alice can retrieve it using her key.

Our key is now $k = (k_{\mathsf{priv}}, k_{\mathsf{pub}}) \in \mathcal{K} = \mathcal{K}_{\mathsf{priv}} \times \mathcal{K}_{\mathsf{pub}}$ which consists of a private key and public key.

Our encryption and decryption functions are now

$$e : \mathcal{K}_{\mathsf{pub}} \times \mathcal{M} \to \mathcal{C}$$
$$d : \mathcal{K}_{\mathsf{priv}} \times \mathcal{C} \to \mathcal{M}$$

$$d(k_{\mathsf{priv}}, e(k_{\mathsf{pub}}, m)) = m$$

We want it to be easy to compute $e_{k_{\mathsf{pub}}}$ and $d_{k_{\mathsf{priv}}}$, but hard to compute $d_{k_{\mathsf{priv}}}$ only knowing k_{pub}.

Something easier to construct, before a full-fledged public key system, is a key exchange:

§6.2 Diffie-Hellman Key Exchange

Q: How can Alice and Bob agree on a secret key over an insecure channel?

Example 6.2

Analogy: A lockbox that can only be used by one person...and both people have to participate to set it up.

Both parties have to agree on a key and have a line of communication before agreeing on a key. This can only be used if both parties are online at the same time.

We start with a prime p and $g \in (\mathbb{Z}/p\mathbb{Z})^\times$ suitably. Alice and bob does the following, all mod p:

Alice	Bob
Generates a	Generates b
$\downarrow$	$\downarrow$
Computes g^a	Computes g^b
Send g^a to Bob	Send g^b to Alice
Computes $(g^b)^a$	Computes $(g^a)^b$

Alice and Bob now know g^{ab}, which is the secret key. Eve, however, only knows g^a and g^b. Alice and Bob can now use this shared secret g^{ab} as a key for symmetric cryptography.

Definition 6.3 (The Diffie-Hellman Problem (DHP))

Given g^a, g^b, calculate g^{ab}.

Remark 6.4. If we can solve the discrete log problem, we can solve the Diffie-Hellman problem.

Vice versa? Can one solve DLP given solution to DHP? *This is unknown*[7].

§6.3 Elgamal Public Key Cryptography

We again start with p prime and $g \in (\mathbb{Z}/p\mathbb{Z})^\times$ suitably. This could be public knowledge, or Alice selects these.

Alice: We have a be Alice's private key, and $A = g^a$ be Alice's public key.

Bob: Has message m he wishes to send. Bob does the following:

1. Generate random k (used only once, to send this message).
2. Compute the following:

[7] There is no known method.

a) $c_1 = g^k \mod p$

b) $c_2 = m \cdot A^k \mod p$

3. Send c_1 and c_2 to Alice.

Alice:

$$(c_1^a) = A^k \text{ so } c_2 \cdot (c_1^a)^{-1} \equiv m \left((g^a)^k\right) \cdot \left((g^a)^k\right)^{-1} \equiv m$$

Basically, they are using Diffie-Hellman key exchange, except g^a is a public key and Bob assumes a secret key, and uses that to encrypt the message and sends it in one go.

§6.3.1 Implementation

We have the following algorithm for encryption and decryption in Elgamal:

```
import ext_gcd, pow_mod
from random import randrange
def e(A, m):
    k = randrange(p)
    return (pow_mod(g, k, p), m * pow_mod(A, k, p))

def d(a, c):
    pow_mod(c[0])
    ...
```

to be continued...

§7 February 11, 2022

§7.1 Elgamal *continued*

Recall: we perform Elgamal by starting with a prime p and $g \in (\mathbb{Z}/p\mathbb{Z})^\times$ which is *public knowledge*.

Alice computes a which is her *private key*, and $A = g^a$ which is her *public key*.

Encryption: Bob generates a random k and sends Alice

$$c_0 \equiv g^k \mod p \qquad c_1 \equiv mA^k \mod p$$

Decryption: Alice computes

$$c_1 \cdot (c_0^a)^{-1} \equiv m(g^a)^k \left((g^k)^a\right)^{-1}$$

We continue as we did from last time:

```
import ext_gcd, pow_mod
from random import randrange
def e(A, m):
    k = randrange(p)
    return (pow_mod(g, k, p), m * pow_mod(A, k, p))

def d(a, c):
    return c[1] * ext_gcd(pow_mod(c[0], a, p), p)[0]
```

Which works as intended (try it out!).

We note a property of Elgamal that there is an expansion factor of 2. It takes *twice* as much space to store c as m. We note that the expansion factor is always at least 1 (otherwise, we wouldn't be able to invert it).

§7.2 Midterm Details

Feb 16 @ 2pm in class. If remote, send email.

Topics will include: *everything up to now* (literally right now).

Focus: More theoretical, less computational. (Both are fair game!)

Resources: Pen/pencil, paper. No notes and no book. Nothing else.

Weighting: 20% Midterm 1 and 30% on Final. 30% Midterm 2, 20% Homework. Half on written and half on in-class exams.

Problem set #3 which is shorter than #2. (Good practice!)

Midterm results/curve will be announced hopefully by Friday after the midterm.

§7.3 Introduction to Group Theory

Groups are an algebraic structure... they're sets endowed with an operation.

Example 7.1

We have that $(\mathbb{Z}/p\mathbb{Z}, +)$ and $((\mathbb{Z}/p\mathbb{Z})^\times, \cdot)$ are both groups.

	$(\mathbb{Z}/p\mathbb{Z}, +)$	$((\mathbb{Z}/p\mathbb{Z})^\times, \cdot)$
Identity:	$0 + a = a$	$1 \cdot a = a$
Inverse:	$a + (-a) = (-a) + a = 0$	$a \cdot a^{-1} = a^{-1} \cdot a = 1$
Associative:	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Commutative:	$a + b = b + a$	$a \cdot b = b \cdot a$

Definition 7.2 (Group)

A group G is a set plus an operation

$$\circ : G \times G \to G$$

satisfying

1. *Identity:* There is $e \in G$ with $e \circ a = a \circ e = a$.
2. *Inverse:* For any $a \in G$, there is $a^{-1} \in G$ with
$$a \circ a^{-1} = a^{-1} \circ a = e$$
3. *Associativity:* $a \circ (b \circ c) = (a \circ b) \circ c$

We additionally say G is Abelian if we have

$$a \circ b = b \circ a$$

Definition 7.3 (Group Order)

The order of G written $\#G$ is the number of elements in group G. If the order is finite, we say G is finite.

Example 7.4

$(\mathbb{Z}/p\mathbb{Z}, +)$ and $((\mathbb{Z}/p\mathbb{Z})^\times, \cdot)$ are both Abelian and finite.

§8 February 14, 2022

§8.1 Groups *continued*

Example 8.1

Some itemize of groups and nongroups:

- $(\mathbb{Z}/N\mathbb{Z}, +)$: Yes (Abelian).
- $(\mathbb{Z}/N\mathbb{Z}, \times)$: No. 0^{-1} does not exist (inverse).
- $((\mathbb{Z}/n\mathbb{Z})^\times, \times)$: Yes (Abelian).
- $(\mathbb{Z} \setminus \{0\}, \times)$: No. 2^{-1} does not exist (inverse).
- $(\mathbb{Z} \setminus \{0\}, +)$: No. No identity e.
- $(\{n \times n \text{ matrices} : \det M \neq 0\}, \times)$: Yes (not Abelian for $n \geq 2$).

Definition 8.2

For $g \in G$, $x = 1, 2, 3, \ldots,$

$$g^x = \underbrace{g \circ g \circ \cdots \circ g}_{x \text{ times}}$$

We extend this to define $g^0 = e$ and $g^{-n} = (g^n)^{-1}$ (so that our usual exponent rules also apply).

Example 8.3

From just now, in $(\mathbb{Z}/N\mathbb{Z}, +)$, $1^3 = 3$.

Definition 8.4 (Element Order)

The smallest (positive) n with $g^n = e$ is called the order of g.

If there is no such n, we say g has infinite order.

Proposition 8.5

If G is a finite group, then every element $g \in G$ has finite order.

Proof. Consider all powers of g

$$g, g^2, g^3, g^4, \ldots$$

so at some point, we will have

$$g, g^2, g^3, g^4, \ldots, g^i, \ldots, g^j, \ldots$$

where g^i and g^j are equal. Then $g^{j-i} = e$. Hence G has a finite order. □

Proposition 8.6

Let $g \in G$ have order k, with $g^n = e$. Then $k \mid n$.

Proof. We use the division algorithm. We write

$$n = q \cdot k + r \text{ with } 0 \leq r < k$$

then we have

$$e = g^n = (g^k)^q g^r = e^q \cdot g^r$$

so $g^r = e$, which forces $r = 0$ since $0 \leq r < k$. So $n = qk \Rightarrow k \mid n$. □

Theorem 8.7

$g^{\#G} = e$. In particular, $\operatorname{ord} g \mid \#G$.

Proof for Abelian groups. Let $G = \{g_1, \ldots, g_n\} = \{gg_1, gg_2, \ldots, gg_n\}$. No two are equal, since we can take inverse of g. We multiply them all together:

$$\begin{aligned} g_1 g_2 \cdots g_n &= (gg_1) \cdots (gg_n) \\ \cancel{g_1 g_2 \cdots g_n} &= \cancel{g_1 \cdots g_n} g^n \\ e &= g^n \end{aligned}$$

so we have as desired. □

This is true even if G is not Abelian - it's Lagrange's Theorem, which we won't cover here[8].

Note that our previous cryptosystems: Diffie-Hellman key exchange and Elgamal, works in *any* group.

Q: Why would we want to be able to pick our group?

Might we want to do this in a group that allows for fast operations? That makes encryption and decryption easy, but it also makes computing the discrete log difficult. We want groups that are *easy enough* and *hard enough*. We might appreciate this by the end of the course...

§8.2 Computation Complexity

How might we quantify "easy" or "hard" in cryptography.

[8] Covered in Math 1530, Abstract Algebra.

Example 8.8

Let $g \in G$ a group. Let's consider exponentiation

$$x \longmapsto g^x$$

if x has k bits (i.e. $x \approx 2^k$). How many steps does it take us to compute g^x? At most $2k$ multiply and add steps.

What about solving the discrete log problem:

$$g^x \longmapsto x$$

where x has k bits. How many steps does this take (naïvely, trying every power)? About 2^k steps.

Definition 8.9 (Big-O)

We say $f(x) = \mathcal{O}(g(x))$ if there are *constants* c and c' with

$$f(x) \leq c \cdot g(x) \quad \text{for all } x \geq c'$$

Example 8.10

Say $f(x) = \mathcal{O}(1) \Leftrightarrow f$ is bounded.

If $f(x) = \mathcal{O}(x^c)$, then we say this is a "easy" problem.[9]

If $f(x) = \mathcal{O}(c^x)$, we think of this as a "hard" problem.

Proposition 8.11

If

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} < \infty$$

then $f(x) = \mathcal{O}(g(x))$.

Proof. Using definition of limits, for any $\varepsilon > 0$:

$$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon \text{ for } x \geq c$$

then $f(x) < (L + \varepsilon) \cdot g(x)$. □

[9] We take x to be the number of bits of the input

Example 8.12
$2x^2 + 5x + 7 = \mathcal{O}(x^2)$.

§9 February 18, 2022

§10 February 23, 2022

§10.1 Chinese Remainder Theorem

Recall: (HW2, Q2) asked us to find x with

$$\begin{aligned} x &\equiv 3 \mod 7 \\ x &\equiv 4 \mod 9 \end{aligned}$$

We solved this by setting

$$\begin{aligned} x = 7y + 3 &\equiv 4 \mod 9 \\ 7y &\equiv 1 \mod 9 \end{aligned}$$

and taking $7^{-1} \pmod 9$ which is 4. So we have

$$\begin{aligned} y &\equiv 4 \mod 9 \\ x = 7y + 3 &\equiv 31 \text{ works} \end{aligned}$$

Theorem 10.1 (Chinese Remainder Theorem)
Let $\{m_i\}$ be a set of pairwise coprime numbers. That is, $\gcd(m_i, m_j) = 1$ for $i = j$. Then the system

$$\begin{aligned} x &\equiv a_1 \mod m_1 \\ x &\equiv a_2 \mod m_2 \\ &\vdots \\ x &\equiv a_n \mod m_n \end{aligned}$$

has a solution.

Proof. By induction on n.

Base case. $n = 1$, then we take $x = a_1$ which is a solution.

Mini inductive step. We first solve for $n = 2$. We have

$$\begin{aligned} x &\equiv a_1 \mod m_1 \\ x &\equiv a_2 \mod m_2 \end{aligned}$$

Let $u_1 \cdot m_1 + u_2 \cdot m_2 = 1$ (by Bezout's identity). Consider quantity

$$\begin{aligned} u_1 m_1 a_2 + u_2 m_2 a_1 &\equiv 0 + 1 \cdot a_1 \equiv a_1 \mod m_1 \\ &\equiv 1 \cdot a_2 + 0 \equiv a_2 \mod m_2 \end{aligned}$$

which solves for $n = 2$.

Full inductive step. Let $n \geq 3$, we solve equation

$$\begin{aligned} x &\equiv a \mod m_1 m_2 \cdots m_{n-1} \\ x &\equiv a_n \mod m_n \end{aligned}$$

where the solution a from the first equation comes from our inductive hypothesis. We can solve this by our mini inductive step (the $n = 2$ case).

Which concludes this proof. □

Example 10.2

The Chinese Remainder Theorem allows us to solve congruences with composite moduli. For example, solve

$$x^2 \equiv 18 \mod 21$$

This is equivalent to solving

$$\begin{aligned} x^2 \equiv 18 = 0 \mod 3 \\ x^2 \equiv 18 = 4 \mod 7 \end{aligned}$$

since $21 = 3 \cdot 7$. So this is equivalent to solving

$$\begin{aligned} x &\equiv 0 \mod 3 \\ x &\equiv \pm 2 \mod 7 \end{aligned}$$

Using CRT, we have

$$1 \cdot 7 + (-2) \cdot 3 = 1$$

so

$$x = 0 \cdot 1 \cdot 7 + (-2) \cdot 2 \cdot 2 = -12 \equiv 9 \mod 21$$

we do check that $9^2 = 81 \equiv 18 \mod 21$.

In general, we claim that square roots mod p are easy to compute.

Proposition 10.3

Let $p \equiv 3 \mod 4$ and a is a square in $\mathbb{Z}/p\mathbb{Z}$ (that is, $x^2 \equiv a$ has a solution).

Then

$$x \equiv a^{\frac{p+1}{4}}$$

is a solution.

Proof. Say $a \equiv b^2 \mod p$. Then

$$(a^{\frac{p+1}{4}})^2 = a^{\frac{p+1}{2}} \equiv (b^2)^{\frac{p+1}{2}} \equiv b^{p+1} \equiv b^{p-1} \cdot b^2 \equiv 1 \cdot b^2 = b^2 \equiv a \mod p$$

which concludes the proof. □

So we can compute square roots modulo a composite number N by taking the prime factor decomposition of N, and taking the square root mod each factor, then using CRT.

Conversely, any efficient algorithm to find square roots mod N can be used to factor N.

Why?

1. We generate an element x mod N.
2. Ask for square root of x^2.
3. Good chance that we get $y \not\equiv \pm x$. We now have
$$x^2 \equiv y^2 \mod N$$
$$(x+y)(x-y) \equiv 0 \mod N$$
4. We can now calculate $\gcd(x+y, N)$ and find some factors of N.

§10.2 Euler's Theorem

Recall: Fermat's Little Theorem which says that

$$a^{p-1} \equiv 1 \mod p \quad \text{for } p \nmid a$$

What happens when we replace p with N where N is composite?

$$a^{N-1} \stackrel{?}{\equiv} 1 \mod N \quad \text{for } \gcd(N, a) = 1$$

No! Recall demo showing counterexample.

Proposition 10.4

If $N = pq$ for primes p and q, then

$$a^{(p-1)(q-1)} \equiv 1 \mod N \quad \text{for } \gcd(N, a) = 1$$

Proof. WLOG taking mod p, we have

$$a^{(p-1)(q-1)} \equiv (a^{p-1})^{q-1} \equiv 1^{q-1} \equiv 1 \mod p$$

Similarly mod q by symmetry. Then it is congruent to $1 \mod pq$. □

We can generalize this...

Proposition 10.5 (Euler's Theorem)

For any composite N, we have

$$a^{\varphi(N)} \equiv 1 \mod N \quad \text{for } \gcd(N, a) = 1$$

§10.3 Exponentiation

Given an exponent x, we can compute e^x fast. Inverting this gives us the *Discrete Log Problem*. Diffie-Hellman Key Exchange and Elgamal rely on this.

What if we think of this as a function of the base? Given a base x, we want to take it to exponent e to get x^e. Inverting this is the *Extracting Roots* problem. This is the basis of the RSA cryptosystem (see next time!).

Claim — Let $\gcd(e, p-1) = 1$. We can construct $de \equiv 1 \mod p-1$. Then $(x^e)^d \equiv x$.

So *extracting roots* is easy mod prime p, but hard mod composites. We'll see this next time.

§11 February 25, 2022

Recall: Midterm discrete log problem where

$$x \to x^e \mod p$$

and a solution we gave was take $(x^e)^p \equiv x$ where $de \equiv 1 \mod (p-1)$.

What if we didn't take this mod p, but instead took it mod pq.

We can take the analog of Fermat's Little Theorem mod pq, where

$$a^{(p-1)(q-1)} \equiv 1 \mod pq$$

What if we did the same thing, instead of taking inverse mod $p-1$, we took it mod $(p-1)(q-1)$ to extract e^{th} roots if we know $(p-1)(q-1)$.

We know p and q, we can easily figure out $(p-1)(q-1)$. Also, if we know $(p-1)(q-1)$, we also know

$$pq - p - q + 1$$

We know that $pq = x$ and $p + q = y$, then pq are roots of $t^2 - yt + x = 0$.

§11.1 RSA Public-Key Cryptography

Alice generates

$$\begin{aligned} p, q &\to \text{Two large prime numbers} \\ e &\to \text{“Public exponent”} \\ N = pq &\to \text{“Public modulo”} \\ \boxed{(e, N)} &\to \text{Public key} \\ (d, N) &\to \text{Private key} \\ \text{where } d \cdot e &\equiv 1 \mod (p-1)(q-1) \end{aligned}$$

Bob has some message m he wishes to send to Alice. Bob sends $m^e \mod N$ and sends it to Alice.

After receiving this message, Alice can recover $(m^e)^d \equiv m \mod N$.

p, q are private, but pq is private. The security of RSA rests on multiplication being easy, but factorization being hard (pq is hard to factorize into p and q).

We might implement such an algorithm like so:

```
from crypto import gcd, ext_gcd, pow_mod
N = p * q
d = ext_gcd(e, (p-1) * (q-1))[0] % ((p-1) * (q-1))
m = 12345678912347869512340102398471237 48
# 0 < m < N is True
c = pow_mod(m, e, N)
pow_mod(c, d, N) # => m
```

If we were Alice and Bob, we *still* have one step to go! We need to find ourselves big prime numbers p, q (finding e is easy, we can just use our gcd algorithm). How do we do that?

§11.2 Primality Testing

Prime hunting!

Prime numbers are reasonably common. So generating a large prime number amounts to generating a number, checking if it's prime, and repeating until we get a prime.

This reduces to the problem of checking if a number is prime. Given an n, is it a prime?

```
from crypto import pow_mod
n = 123874610239487102893741890237023
pow_mod(2, n−1, n)
```

gives us a basic primality check. Fermat's Little Theorem says that if n is prime, then $2^{p-1} \equiv 1 \mod n$.

Definition 11.1 (Witnesses)

We say that a is a witness for the compositeness of n if

$$a^{n-1} \not\equiv 1 \mod n$$

and $gcd(a, n) = 1$.

We have a problem! Fermat's Little Theorem is not an if-and-only-if. We can have numbers that pass this test for almost every base. Take $n = 3 \cdot 11 \cdot 17 = 561$.

```
from crypto import pow_mod, gcd
for a in range(n):
    if gcd(a, n) == 1:
        print(pow_mod(a, n−1, n))
```

gives 1 for *everything*[10]

Definition 11.2 (Carmichael Number)

A Carmichael number is a composite number with *no* witness of compositeness.

Example 11.3

561 is a Carmichael Number.

We now *almost* have a way of checking for primality, but it doesn't always quite work.

[10] oh no!

Since taking to the power of $n-1$ is a group homomorphism, then Lagrange's Theorem states that if there is a witness, then there are *a lot* of witnesses.

Proposition 11.4

Let p be an odd prime. Write

$$p-1 = 2^k \cdot q \quad \text{with } q\text{-odd}$$

Then either

$$a^q \equiv 1 \mod p$$

or one of $a^q, a^{2q}, a^{4q}, \ldots, a^{2^{k-1}q}$ is $\equiv -1 \mod p$.

Proof. We look at this sequence

$$a^q, a^{2q}, a^{4q}, \ldots, \underbrace{a^{2^i q}}_{\not\equiv 1 \pmod p}, \underbrace{a^{2^{i+1}q}}_{\equiv 1 \pmod p}, \ldots, a^{2^{k-1}q}, a^{2^k q}$$

We have that $a^{2^k q} \equiv 1 \mod p$ by Fermat's Little Theorem. There's some point where we 'become' congruent to $1 \mod p$. If the first one is one, then we have the first case. Otherwise we have $a^q \not\equiv 1 \mod p$, then we repeatedly square until we get to $1 \mod p$. Then *right before* we turned to $1 \mod p$, we would have had $-1 \mod p$. In our example above, this is $a^{2^i q}$. □

§12 February 28, 2022

§12.1 Miller-Rabin Primality Test

Recall proposition 11.4 from last class.

Proposition

Let p be an odd prime. Write

$$p-1 = 2^k \cdot q \quad \text{with } q\text{-odd}$$

Then either

$$a^q \equiv 1 \mod p$$

or one of $a^q, a^{2q}, a^{4q}, \ldots, a^{2^{k-1}q}$ is $\equiv -1 \mod p$.

For 561, we write $561 = 2^4 \cdot 35$.

$$\begin{aligned} 2^{35} &\equiv 263 \pmod{561} \\ 2^{70} &\equiv 166 \pmod{561} \\ 2^{140} &\equiv 67 \pmod{561} \\ 2^{280} &\equiv 1 \pmod{561} \end{aligned}$$

using the following code...

```
from crypto import pow_mod

pow_mod(2, 35) # 263
263 ** 2 % 561 # 166
166 ** 2 % 561 # 67
67 ** 2 % 561 # 1
```

Definition 12.1 (Miller-Rabin Witness)

a is a Miller-Rabin witness if a does not satisfy above proposition.

Theorem 12.2

If n is composite, then at at least 75% of $a \in (\mathbb{Z}/n\mathbb{Z})$ are Miller-Rabin witnesses.

Proof. Given on faith. □

Algorithm 12.3 (Miller-Rabin Probabilistic Primality Test) —

```
from random import randrange
from crypto import pow_mod

def miller_rabin(n, a):
    """
    Miller–Rabin primality test on number n and base a
    """
    q, k = n − 1, 0
    # Write n − 1 = 2^k · q
    while q % 2 == 0:
        k = k + 1
        q = q // 2
    a = pow_mod(a, q, n)
    if a == 1 or a == n − 1:
        return False
    for _ in range(k − 1):
        a = a ** 2 % n
        if a == n − 1:
            return False
    return True
```

```
def is_prime(n):
    for _ in range(50):
        if miller_rabin(n, randrange(1, n)):
            return False
    return True
```

Using this, we can find large prime numbers using

```
from crypto import miller_rabin, is_prime
def next_prime(n):
    while not is_prime(n):
        n = n + 1
    return n
```

Theorem 12.4 (Prime Number Theorem)

Probability that n is prime is about

$$\frac{1}{\log(n)}$$

More formally,

$$\lim_{x\to\infty} \frac{\#\text{ of primes} \leq x}{x/\log x} = 1$$

So the Miller-Rabin test lets us efficiently find (large) prime numbers.

§13 March 7, 2022

§13.1 Pollard's $p-1$ Method

Recall: the Pollard's $p-1$ method from last time. Let

$$N = p \cdot q$$

$$\begin{matrix} p-1 \mid n! \\ q-1 \nmid n! \end{matrix} \to \gcd(N, a^{n!}-1) = p$$

```
from pollard import *

def factor(N):
    # Naive, brute force factoring algorithm.
    i = 2
    while N % i != 0:
        i += 1
    return i
```

```
9
10 def factor(N, a=2):
11     # Pollard's method
12     i = 1
13     while gcd(a − 1, N) == 1:
14         i += 1
15         a = pow_mod(a, i, N)
16     return gcd(a − 1, N)
```

This algorithm is *significantly* faster than random guesses. By how much? We can formalize this intuition of size of prime factors:

Definition 13.1

We say n is B-smooth if all prime factors are $\leq B$. We define

$$\Psi(X, B) = \#\text{ of } B\text{-smooth } n \leq X.$$

Theorem 13.2

Let X, B increase together. Suppose

$$(\log X)^{\varepsilon} \leq \log B \leq (\log X)^{1-\varepsilon}$$

Then

$$\frac{\Psi(X,B)}{X} = u^{-u\cdot(1+o(1))} \quad \text{where } u = \frac{\log X}{\log B}.$$

What is $o(1)$?

Definition 13.3

Say $f(x) = o(g(x))$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

So "$o(1)$" is to mean something whose limit is 0. This is in contrast to $O(1)$ which means something whose limit is finite.

For our purposes, we say that the probability X is B-smooth is $\approx u^{-u}$ where $u = \frac{\log X}{\log B}$.

What if we do Pollard $p-1$ for $e^{\sqrt{\log p}}$ steps? The probability of success is then

$$\sqrt{\log p}^{-\sqrt{\log p}} \approx e^{-\frac{1}{2}\log\log p\cdot\sqrt{\log p}} \gg p^{-\varepsilon}$$

By brute force, doing p^{ε} steps gives a probability of success $\approx p^{\varepsilon-1}$.

§13.2 Quadratic Sieve

Goal: find a, b with $a^2 \equiv b^2 \pmod{N = pq}$, hence $(a-b)(a+b) \equiv 0 \mod N$. $\gcd(a+b, N)$ will allow us to recover p or q with decent probability.

Example 13.4

For example, if

$$\begin{aligned}
(\lfloor\sqrt{N}\rfloor + 1)^2 &\equiv (\lfloor\sqrt{N}\rfloor + 1)^2 - N = 2^1 \cdot 3^1 \\
(\lfloor\sqrt{N}\rfloor + 2)^2 &\equiv (\lfloor\sqrt{N}\rfloor + 2)^2 - N = (\text{not smooth}) \\
(\lfloor\sqrt{N}\rfloor + 3)^2 &\equiv (\lfloor\sqrt{N}\rfloor + 3)^2 - N = 2^2 \cdot 3^1 \\
(\lfloor\sqrt{N}\rfloor + 4)^2 &\equiv (\lfloor\sqrt{N}\rfloor + 4)^2 - N = 2^1 \cdot 3^2 \\
&\vdots
\end{aligned}$$

then we have

$$\left[(\lfloor\sqrt{N}\rfloor + 1)^3\right]^2 \equiv \left[(\lfloor\sqrt{N}\rfloor + 3)(\lfloor\sqrt{N}\rfloor + 4)\right]^2$$

Which comes from the fact that the exponent vectors

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$$

are linearly dependent. We can take these mod 2 (the parity) since we square. We can also rewrite this as[a]

$$\left[(\lfloor\sqrt{N}\rfloor + 1)(\lfloor\sqrt{N}\rfloor + 3)(\lfloor\sqrt{N}\rfloor + 4)\right]^2 \equiv \left(2^2 \cdot 3^2\right)^2$$

[a] Using the definition of linear dependence

So we can do the following steps:

1. Pick smoothness bound B.
2. Find integers
$$(\lfloor\sqrt{N}\rfloor + i)^2 - N$$
that are B smooth.
3. Find linear relationship between exponent vectors. Then we get a congruence $a^2 \equiv b^2 \pmod{N}$ after which we can try to find factors of N.

§14 March 9, 2022

***(missed?)

§15 March 11, 2022

§15.1 Quadratic Sieve *continued*

We resume our discussion of the example of example 13.4. We want to find an optimal B for the algorithm, and we do this by analyzing runtimes.

The runtime of X is first approximately $B \cdot u^u$ where

$$u = \frac{\log \sqrt{N}}{\log B}.$$

And in addition, solving a $B \cdot B$ system of linear equations. It takes B^2 operations to zero out one column (B rows and subtracting a column takes B operations), then we do this for B columns. So the runtime is B^3.

We now have u^u is decreasing in B and B^3 is increasing in B. Minimizing the runtime, we set $B^3 \sim B \cdot u^u$, i.e. $B^2 \sim u^u$. Using very sketchy mathematics,

$$\begin{aligned} u^u &\sim B \\ u \log u &\sim B \\ \frac{logN}{\log B} \sim u &\sim B \\ \log B &\sim \sqrt{\log N} \\ B &\sim e^{\sqrt{\log N}} \\ u \sim \frac{\log \sqrt{N}}{\log B} &\sim \frac{\log N}{\log B} \sim \sqrt{\log N} \end{aligned}$$

is a *loose* guess. But we can use this to make a more approximate guess.

Being more rigorous, $u^u = B^2$ gives $u \log u = 2 \log B$. Then using our estimate for u from above, we have

$$\begin{aligned} u \cdot \log \left[\sqrt{\log N}\right] &= 2 \log B \\ \frac{1}{2} \frac{\frac{1}{2} \log N}{\log B} \log \log N = \frac{1}{2} u \log \log N &= 2 \log B \\ (\log B)^2 &= \frac{1}{8} \log N \log \log N \\ \Rightarrow B &\sim e^{\sqrt{\frac{1}{8} \log N \log \log N}} \end{aligned}$$

where we note the difference of a factor of $\frac{1}{8} \log \log N$ isn't that far off from $e^{\sqrt{\log N}}$. So total runtime is around B^3 which is $e^{\sqrt{\frac{9}{8} \log N \log \log N}}$. *It's not super fast but not totally stupid.*

Realistically, the B^3 can be reduced to B^2 in solving our $B \times B$ system, but we can get rid of our factors of 2's from before. This lets us get rid of the factor of $\frac{9}{8}$ so our total runtime is like

$$e^{\sqrt{\log N \log\log N}}$$

There are even faster algorithms, namely the number field sieve. It replaces the square root from above into a cube root and has runtime

$$e^{\sqrt[3]{c \cdot \log N (\log\log N)^2}}$$

§15.2 Index Calculus & Discrete Logs

The discrete log problem is solving x given $g^x = a$ and we know g and a. We can do a similar strategy to the quadratic sieve. We calculate

$$\begin{aligned} g^1 &\pmod p = 2 \\ g^2 &\pmod p = (\text{not smooth}) \\ g^3 &\pmod p = 2 \cdot 3 \end{aligned}$$

where we pick some smoothness bound B. We then find $g^i \equiv (\text{B-smooth}) \mod p$. We find enough B smooth things such that by linear algebra, we can solve $g^x = 2, 3, 5, 7, \ldots$.

We take

$$\begin{aligned} a &\mod p = (\text{not smooth}) \\ a \cdot g^{-1} &\mod p = (\text{not smooth}) \\ a \cdot g^{-2} &\mod p = 3 \end{aligned}$$

which is smooth. So we find $a \cdot g^{-i} \equiv (\text{B-smooth}) \mod p$. Then by linear algebra we can solve for a given lots of smooth g^i.

Question. What is the runtime of this?

We need to go up to g^x where x is about $B \cdot u^u$ (u^u is the probability of being B-smooth). Linear algebra will take runtime B^2. Similar to just now, this is exactly the same problem as above except $\sqrt{N}$ is replaced with P. Before (in the quadratic sieve), we had

$$B \sim \exp\left(\frac{1}{4} \log N \log\log N\right)$$

and

$$\text{Runtime} \sim \exp\left(\log N \log\log N\right)$$

Replacing $\log N$ with $2 \log P$, our bound becomes

$$B \sim \exp\left(\frac{1}{2} \log p \log\log p\right)$$

$$\text{Runtime} \sim \exp\left(2 \log p \log \log p\right)$$

What if $g \in$ subgroup of order q? Babystep-Giantstep changes from $\sqrt{p}$ to $\sqrt{q}$. With index calculus, there is no advantage of knowing that g is im a smaller subgroup. We don't have a decrease in runtime.

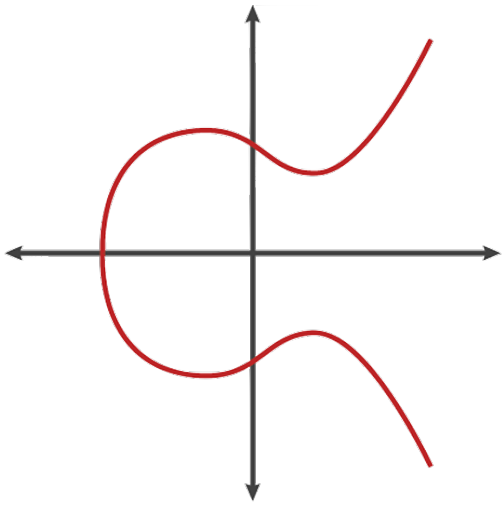
§16 March 14, 2022

§16.1 Elliptic Curves

An elliptic curve is an equation of the form

$$y^2 = x^3 + ax + b$$

It looks like this:



What's special about cubic equations? We have a special property that every line L meets E in 3 points.

What happens when we have a tangential line? We count multiplicity.

What happens when it only meets at 1 or 2 points? We count complex roots.

What happens with vertical lines that only meet at 2 points? We include $\mathcal{O} =$ "point at ∞".

Where does $\mathcal{O}$ come from? It comes from the $\mathbb{RP}^2$ (the real projective plane) or $\mathbb{CP}^2$ (the complex projective plane).

Given two points A and B, we can get a third point C which is the third point on the line passing through A and B. Taking this as a binary operation...does this give us a group?

Consider:

$$A + B = C$$
$$A + C = B$$
$$A + A = \mathcal{O}?$$

Maybe we can declare

$$A + B + C = \mathcal{O}$$

If we call the reflection of C across the x-axis is D, we have

$$A + B + C = \mathcal{O}$$
$$C + D + \mathcal{O} = \mathcal{O}$$
$$A + B = D$$

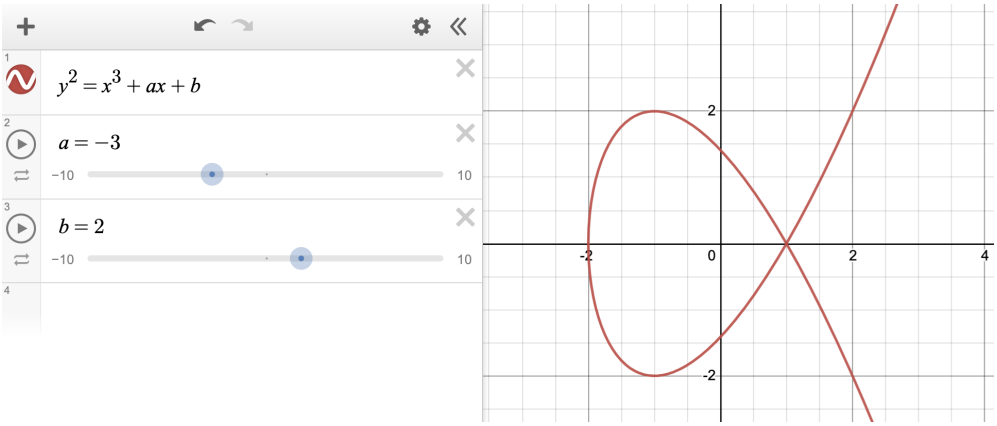
So we have that the group law is $A + B$ is the reflection of the third point C across the x-axis.

Definition 16.1 (Elliptic Curve)
An elliptic curve is the set of solutions to

$$y^2 = x^3 + ax + b$$

plus a point $\mathcal{O}$ at infinity...where a, b satisfy $4a^3 + 27b^2 \neq 0$.

Recall from high school that $ax^2 + bx + c$ gives discriminant $\Delta = b^2 - 4ac$. Taking a cubic equation $x^3 + ax + b$, the discriminant is $\Delta = -16(4a^3 + 27b^2)$. This is also saying $x^3 + ax + b$ has no repeated roots. For it to be tangent to the x-axis, it has to self-intersect. But every line passing through the intersection is a tangent line. Messy messy things happen:



We take the fact that an elliptic curve is a group on faith, with the group operation defined above.

§16.2 Addition on Elliptic Curves

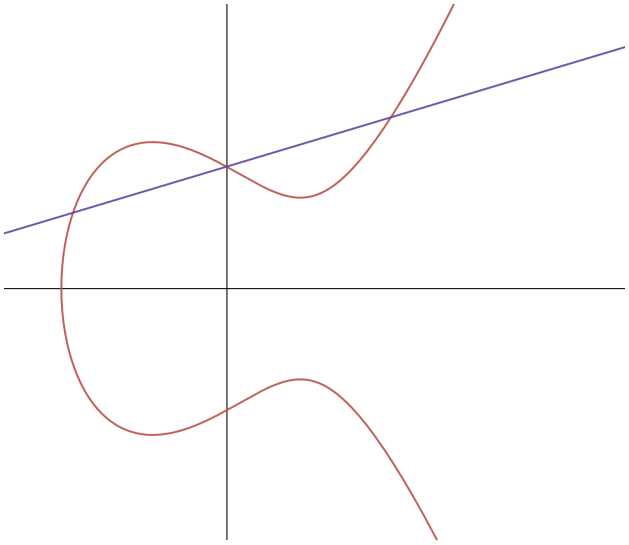
- $P + \mathcal{O} = \mathcal{O} + P = P$. That is, $\mathcal{O}$ is the identity.
- If for points $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ and $x_1 = x_2$, $y_1 = -y_2$. Then $P_1 + P_2 = \mathcal{O}$.
- If $P_1 \neq P_2$, $\lambda =$ Slope of $L = \frac{y_2 - y_1}{x_2 - x_1}$. So the equation of L is $y - y_1 = \lambda(x - x_1)$. (We find the third point and reflect it). Will pick up here on Wednesday.

§17 March 16, 2022

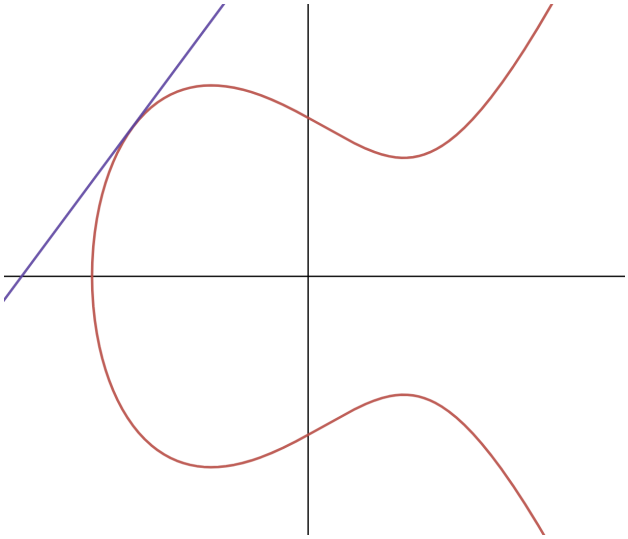
§17.1 Addition on Elliptic Curves *continued*

We said last time that we had some rules:

- $P + \mathcal{O} = \mathcal{O} + P = P$.
- If $x_1 = x_2$ and $y_1 = -y_2$, we have
$$P_1 + P_2 = \mathcal{O}$$
- In other cases, we need to calculate the slope of the line.



If $P_1 \neq P_2$, the slope of L is
$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}$$



If $P_1 = P_2$, then the slope would be the tangent:

$$\begin{aligned} y^2 &= x^3 + ax + b \\ 2y \cdot dy &= (3x^2 + a) \cdot dx \\ \frac{dy}{dx} &= \frac{3x^2 + a}{2y} \end{aligned}$$

so we have

$$\lambda = \frac{3x^2 + a}{2y}$$

We want to solve systems

$$\begin{cases} y^2 = x^3 + ax + b \\ y = y_1 + \lambda(x - x_1) \end{cases}$$

which gives us

$$\begin{aligned} [y_1 + \lambda(x - x_1)]^2 &= x^3 + ax + b \\ 0 &= x^3 - \boxed{\lambda^2} x^2 + (a + 2\lambda_1 x_1)x + b - y_1 - \lambda x_1^2 \\ &= (x - x_1)(x - x_2)(x - x_3) \\ &= x^3 - \boxed{(x_1 + x_2 + x_3)} x^2 + (x_1x_3 + x_2x_3 + x_1x_2)x - x_1x_2x_3 \end{aligned}$$

With some working out (taking the coefficient of x^2),

$$\boxed{x_3 = \lambda^2 - x_1 - x_2}$$

and $-y_3 = y_1 + \lambda(x_3 - x_1)$ so we have

$$\boxed{y_3 = -y_1 - \lambda(x_3 - x_1)}$$

are our points by addition where

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}$$

or when $P_1 = P_2$,

$$\lambda = \frac{3x_1^2 + a}{2y_1}.$$

§17.2 Elliptic Curves over Finite Fields

Our definition stays the same, except x, y are elements of $\mathbb{Z}/p\mathbb{Z}$. How do we add points? We could do it geometrically, but setting this up is outside the scope of this class...

For addition, we use the same formulas that we've derived for x_3 and y_3, and they still make perfect sense mod p. *For whatever reasonable notion of geometry we have over $\mathbb{Z}/p\mathbb{Z}$, they work with these formulas.*

Example 17.1

We take elliptic curve

$$y^2 = x^3 + x + 2 \quad \text{over } \mathbb{Z}/5\mathbb{Z}.$$

How do we find elements in this elliptic curve? We can try them all.

- If $x = 0$, $y^2 = 2$, of which there are no solutions.
- If $x = 1$, $y^2 = 4$, of which $y = 2, 3$ are solutions.
- If $x = 2$, $y^2 = 2$ again, of which there are no solutions.
- If $x = 3$, $y^2 = 2$, of which there are no solutions.
- If $x = 4$, $y^2 = 0$, so $y = 0$ is one solution.

We have $(1,2), (1,3), (4,0), \mathcal{O}$ are the elements of this elliptic curve. We have these

$+$	$\mathcal{O}$	$(1,2)$	$(4,0)$	$(1,3)$
$\mathcal{O}$	$\mathcal{O}$	$(1,2)$	$(4,0)$	$(1,3)$
$(1,2)$	$(1,2)$	$(4,0)$	$(1,3)$	$\mathcal{O}$
$(4,0)$	$(4,0)$	$(1,3)$	$\mathcal{O}$	$(1,2)$
$(1,3)$	$(1,3)$	$\mathcal{O}$	$(1,2)$	$(4,0)$

Let's implement this:

```
O = "the point O"
def add(P1, P2, a, p):
    if P1 == O:
        return P2
    if P2 == O:
        return P1
    x1, y1 = P1
```

```
    x2, y2 = P2
    if x1 == x2 and (y1 + y2) % p == 0:
        return O
    if P1 == P2:
        lam = (3 * x1**2 + a) * ext_gcd(2 * y1, p)[0] % p
    else:
        lam = (y2 - y1) * ext_gcd(x2 - x1, p)[0] % p
    x3 = (lam**2 - x1 - x2) % p
    y3 = (lam * (x1 - x3) - y1) % p
    return x3, y3
```

§18 March 18, 2022

§18.1 Elliptic Curves over Finite Fields *continued*

Recall: from last class, we had our toy example

Example

We take elliptic curve

$$y^2 = x^3 + x + 2 \quad \text{over } \mathbb{Z}/5\mathbb{Z}.$$

with group law

$+$	$\mathcal{O}$	$(1,2)$	$(4,0)$	$(1,3)$
$\mathcal{O}$	$\mathcal{O}$	$(1,2)$	$(4,0)$	$(1,3)$
$(1,2)$	$(1,2)$	$(4,0)$	$(1,3)$	$\mathcal{O}$
$(4,0)$	$(4,0)$	$(1,3)$	$\mathcal{O}$	$(1,2)$
$(1,3)$	$(1,3)$	$\mathcal{O}$	$(1,2)$	$(4,0)$

We define some more useful functions:

```
def minus(P, p):
    if P == O:
        return O
    else:
        x, y = P
        return (x, (-y) % p)
```

What about multiplication? We can repeatedly add:

```
def multiply(P, n, a, p):
    S = O
    for _ in range(n):
        S = add(P, S, a, p)
    return S
```

but this might be very slow (it does n iterations). We can do something similar to fast powering for exponentiation, except we repeatedly double our point:

```
def multiply(P, n, a, p):
    S = 0
    while n != 0:
        if n % 2 == 1:
            S = add(S, P, a, p)
        n = n // 2
        P = add(P, P, a, p)
    return S
```

Question. What is the order of $E(\mathbb{F}_p)$? That is, how many points are there on the elliptic curve?

Let's say we take $x, y, \ldots$. We want to solve

$$y^2 \stackrel{?}{=} x^3 + ax + b$$

There are p^2 different (x, y). The probability that this equality holds is *like* $\frac{1}{p}$. So there are about $p^2 \cdot \frac{1}{p} + 1 \approx p + 1$ (added one for the point $\mathcal{O}$) elements in $E(\mathbb{F}_p)$.

Theorem 18.1

$$|(p+1) - \#E(\mathbb{F}_p)| \leq 2\sqrt{p}.$$

That is, the difference between our estimate $p+1$ and the actual number of points in $E(\mathbb{F}_p)$ is bounded by $2\sqrt{p}$.

Proof. (Beyond the scope of this class.) □

Remark 18.2.

1. We note that this number can be efficiently computed (in polynomial time in the digits of p). *(Again, beyond the scope of this class.)*
2. We call this number $|(p+1) - \#E(\mathbb{F}_p)|$ the trace of Frobenius. *(You guessed it: again, beyond the scope of this class.)*

Just for funsies, we can compute the *trace of Frobenius*[11]:

```
p = next_prime(92834712736591432)
E = EllipticCurve([34123498, GF(p)(2349182347)])
E.trace_of_frobenius()
```

[11] Using *Sage*.

§18.2 Elliptic Diffe-Hellman Key Exchange

§18.2.1 Elliptic Discrete Log Problem

This is based on the *Elliptic Discrete Log Problem*: Given E an elliptic curve over $\mathbb{F}_p$ with P a point on E. We take P, n and calculate

$$n \cdot P$$

The elliptic curve discrete log problem is given point $n \cdot P$, computing n.

The best known algorithm for *ECDLP* is Babystep-Giantstep, which runs in $\mathcal{O}(\sqrt{p})$ and $\mathcal{O}(\sqrt{p})$ memory. There is *no* analog of index calculus. This could be good or bad... This could mean a lack of knowledge about elliptic curves. We could also create greater security at smaller key sizes.

§18.2.2 Sharing Secrets

Public information: we have some p prime and E an elliptic curve over $\mathbb{F}_p$. We have P a point in the elliptic curve E.

Alice and Bob do the following:

1. Alice and Bob each generate a and b. Alice computes $a \cdot P$ and Bob computes $b \cdot P$. This is shared to each other (and public).

2. Alice now computes $a \cdot (b \cdot P)$ and Bob computes $b \cdot (a \cdot P)$. These are all equal to $(a \cdot b) \cdot P$ which is a shared secret.

§19 March 21, 2022

§19.1 Elliptic Curve Elgamal

As usual, we have some public knowledge, private key, and public key. The idea is to replace multiplication in $\mathbb{F}_p^\times$ with addition on E.

Public Knowledge:
p — prime.
E — elliptic curve over $\mathbb{F}_p$.
$P \in E(\mathbb{F}_p)$ — point.

Private Key:
n — private key.

Public Key:
$Q = n \cdot P$ — public key.

Encrpytion:
Bob has a message $M \in E(\mathbb{F}_p)$.

1. Choose random k.

2. Compute

$$C_1 = k \cdot P$$
$$C_2 = M + k \cdot Q$$

   Send (C_1, C_2) to Alice.

Decryption:
Alice will compute

$$C_2 - n \cdot C_1 = M + k \cdot Q - nk \cdot P = M$$

We now implement this:

```
from ec import ext_gcd, add, minus, multiply

# Private key for Alice
n = randrange(q)

# Public key for Alice
Q = multiply(P, n, a, p)

def e(Q, M):
    """Encryption Function"""
    k = randrange(q)
    c1 = multiply(P, k, a, p)
    c2 = add(M, multiply(Q, k, a, p), a, p)
    return (c1, c2)

def d(n, C):
    """Decryption Function"""
    c1, c2 = C
    return add(c2, minus(multiply(c1, n, a, p), p), a, p)
```

The expansion factor is 2. Even if we think of putting our message into only the x coordinate, the y coordinate is determined by the x coordinate so the factor is still 2.

§19.2 Elliptic Curve DSA

Since we have Elgamal, we can also have DSA with Elliptic Curves.

Public Knowledge & Public Key:
Same as Elgamal
p — prime.
E — elliptic curve over $\mathbb{F}_p$.
$Q = n \cdot P$ — public key.

Private Key:
n — private key.

Signing:
Alice has document $d \in \mathbb{Z}/q\mathbb{Z}$.

1. Choose random k.

2. Compute $(x, y) = k \cdot P$.

$$\begin{aligned} s_1 &= x \\ s_2 &= (d + ns_1) \cdot k^{-1} \pmod{q} \end{aligned}$$

Verification:
We can verify the signature as follows:

$$\begin{aligned} v_1 &= d \cdot s_2^{-1} \pmod{q} \\ v_2 &= s_1 s_2^{-1} \pmod{q} \end{aligned}$$

$$\begin{aligned} v_1 \cdot P + v_2 \cdot Q &= ds_2^{-1} \cdot P + s_1 s_2^{-1} k \cdot P \\ &= (d + s_1 n) s_2^{-1} P \\ &= kP \end{aligned}$$

(x-coord of $v_1 P + v_2 Q) = s_1$ check to verify signature.

Again, we can implement this:

```
from ec import *

def sign(n, d):
    """Signing document d with private key n"""
    k = randrange(q)
    x, _ = multiply(P, k, a, p)
    s1 = x
    s2 = ((d + n * s1) * ext_gcd(k, q)[0]) % q
    return (s1, s2)

def verify(Q, d, s):
    """Verifies document d's signature s with public key Q"""
    s1, s2 = s
```

```
    v1 = (d * ext_gcd(s2, q)[0]) % q
    v2 = (s1 * ext_gcd(s2, q)[0]) % q
    return add(multiply(P, v1, a, p), multiply(Q, v2, a, p), a, p)[0] == s1
```

Remark 19.1. The specific numbers used in lecture is the elliptic curve used in the Bitcoin blockchain. Being able to forge signatures in this elliptic curve is to topple the Bitcoin market.

§20 March 23, 2022

§20.1 Elliptic Curve Factorization

Recall: Pollard $p-1$ method.

The idea to factor N was we take $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ and we compute $\gcd(N, a^{n!}-1)$ where $(p-1) \mid n!$ but $(q-1) \nmid n!$.

The idea for elliptic curves is to take $P \in E(\mathbb{Z}/N\mathbb{Z})$ and we calculate $n! \cdot P$.

Let's first fix our code for composite N:

```
def invert(a, n):
    if gcd(a, n) == 1:
        return ext_gcd(a, n)[0]
    else:
        print("OH NO!!!!!", gcd(a, n))

def add(P1, P2, a, p):
    if P1 == O:
        return P2
    if P2 == O:
        return P1
    x1, y1 = P1
    x2, y2 = P2
    if x1 == x2 and (y1 + y2) % p == 0:
        return O
    if P1 == P2:
        lam = (3 * x1**2 + a) * invert(2 * y1, p) % p
    else:
        lam = (y2 - y1) * invert(x2 - x1, p) % p
    x3 = (lam**2 - x1 - x2) % p
    y3 = (lam * (x1 - x3) - y1) % p
    return x3, y3
```

Example 20.1

We use elliptic curve $E : y^2 = x^3 + 3x + 7 \mod 187$. We calculate

$$5 \cdot (38, 112)$$

which fails, since we weren't able to find the inverse of something mod 187 which wasn't coprime. However, we figured out a factor of 187, which is what we wanted. We found 11 which is a prime factor of 187.

We can, however, do this modulo each of the prime factors of 187, 11 and 17. We find that

$$\begin{aligned} 5 \cdot (38, 112) &= \mathcal{O} && \text{in } \mathbb{F}_{11} \\ 5 \cdot (38, 112) &= (13, 13) && \text{in } \mathbb{F}_{17} \end{aligned}$$

What happened was that we got $\mathcal{O}$ mod 11 and got not $\mathcal{O}$ mod 17.

In the elliptic curve method, we win if $\#E(\mathbb{F}_p) \mid n!$ but $\#E(\mathbb{F}_q) \nmid n!$ and so

$$\begin{aligned} n! \cdot P &= \mathcal{O} \pmod{p} \\ &\neq \mathcal{O} \pmod{q} \end{aligned}$$

We have $\#\mathbb{F}_p^\times = p - 1 \approx p$ and $\#E(\mathbb{F}_p) \approx p$, so we have about the same 'chance of winning'.

Question. What is the probability of success per unit time?

We calculate $n! \cdot P$ takes about $n \log n \approx n$ time. The probability of success is u^{-u} where $u = \frac{\log p}{\log n}$. So the probability of success per unit time if $\frac{1}{n \cdot u^u}$.

What minimizes $n \cdot u^u$? When $n \approx u^u$.

Our crude estimate is about

$$\begin{aligned} \log n \approx u \log u \approx u &= \frac{\log p}{\log n} \\ (\log n)^2 &\approx \log p \\ n &\approx e^{\sqrt{\log p}} \end{aligned}$$

and our refined estimation is about

$$\begin{aligned} \log n \approx u \log u &\approx u \log \left(\frac{\log p}{\sqrt{\log p}} \right) \\ &\approx \frac{1}{2} \log \log p \cdot \frac{\log p}{\log n} \\ n &\approx e^{\sqrt{\frac{1}{2} \log p \log \log p}} \end{aligned}$$

so our probability of success per unit time is

$$\approx \frac{1}{n \cdot u^u} \approx \frac{1}{n^2} \approx e^{-\sqrt{2 \log p \log \log p}}$$

With Pollard's $p-1$ method, our probability of success will increase then decrease. With the elliptic curve method, we can optimize our probability of success by simply picking a new elliptic curve at the same n.

This is the best known method for finding small prime factors. The runtime depends on the size of the prime factor we're trying to find instead of the number itself.

§21 March 25, 2022

§21.1 Quantum Computation

Warm-up: Deutch's Algorithm

Setup: We have a function $f : \{0,1\} \to \{0,1\}$. We note there are 4 possibilities for functions f. Let's say we have a black box that takes x and computes $f(x)$, taking an hour to run.

Our question is "is f constant"? Does $f(0) = f(1)$?

Classically, the fastest algorithm takes 2 hours. We pass in 0 and then pass in 1 and check if they are equal.

At the end, we know $f(0) \stackrel{?}{=} f(1)$. But we know more! We know exactly what $f(0)$ and $f(1)$ are.

Quantum Reformulation: we can *manipulte quantum states*. We consider the simplest system, an electron e^-. An electron has a property called *spin*: which can be in one of two states:

$$|\uparrow\rangle, \quad |\downarrow\rangle$$

which we notate 0 and 1.

> **Remark.** The state of the system is **not a "probabilistic classical state".**. It's **not** represented as
>
> $$a \cdot |\uparrow\rangle + b \cdot |\downarrow\rangle$$
>
> where a, b are positive reals with $a + b = 1$.

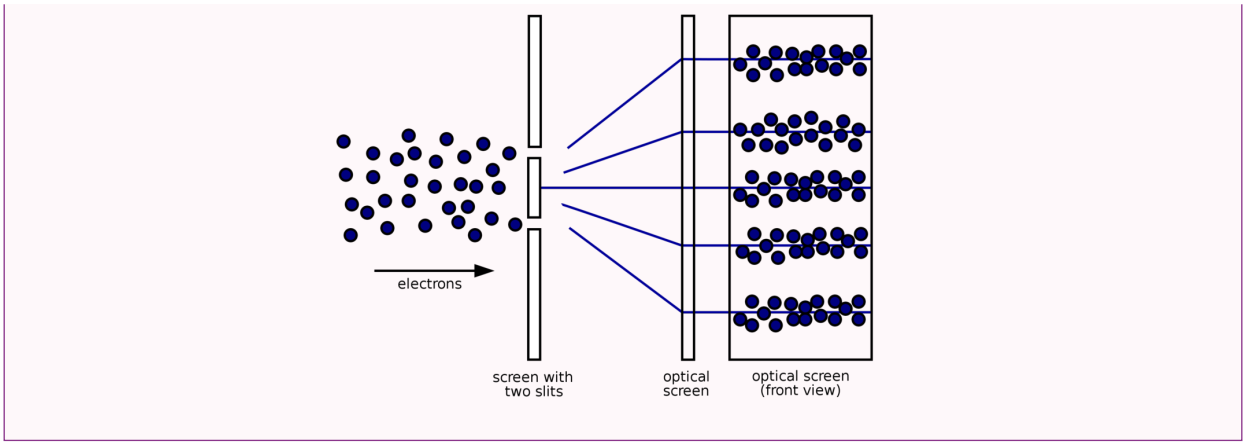
The *actual state* of our machine is

$$a \cdot |\uparrow\rangle + b \cdot |\downarrow\rangle$$

where $a, b \in \mathbb{C}$ where $|a|^2 + |b|^2 = 1$. The probability of measuring $|\uparrow\rangle$ is $|a|^2$ and $|\downarrow\rangle$ is $|b|^2$.

Example 21.1

The typical experiment for this is the light passing through two slits:



We'll run black box again with quantum mechanics - except we only run it once and don't require the knowledge of what $f(0)$ and $f(1)$ *actually are*.

The laws of quantum mechanics are time symmetric. Thus,

$$f : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 0 \end{cases}$$

cannot exist in a quantum computer since it's not invertible. How do we encapsulate our function to be run in a quantum computer?

We define a new function

$$F(x, y) = (x, f(x) + y \bmod 2)$$

F and f encode exactly the same information. But F is defined in a way such that it fits on a quantum computer since it's an invertible function. Classically, it makes no difference having a box that computes F or a box that computes f. Computing one gives us the other.

§21.1.1 Deutch's Problem

Given a *quantum* black box for F. Can we determine whether $f(0) = f(1)$?

The idea is that we feed in x and y as superpositions between 0 and 1 and exploit cancellation. We evaluate

$$\begin{aligned} &F\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \\ &= \frac{1}{2}|0\rangle \otimes |f(0)\rangle - \frac{1}{2}|0\rangle \otimes |f(0)+1\rangle + \frac{1}{2}|1\rangle \otimes |f(1)\rangle - \frac{1}{2}|1\rangle \otimes |f(1)+1\rangle \\ &= \frac{1}{2}\boxed{\left((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right)} \otimes (|0\rangle - |1\rangle) \end{aligned}$$

What if I measure the first register? We get 50% chance of $|0\rangle$ and 50% chance of $|1\rangle$ and we gain no information.

Graphing this, we have 4 states $(1,1)$, $(-1,-1)$, $(1,-1)$, and $(-1,1)$. Applying a rotation matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

we get states on the axes. So we get $|1\rangle$ if $f(0) = f(1)$ and $|0\rangle$ if $f(0) \neq f(1)$.

§22 April 4, 2022

§22.1 Quantum Computing *continued*

Recall: Deutch's Algorithm. Given black box that implements $f : \{0,1\} \to \{0,1\}$. We define $F : \{0,1\}^2 \to \{0,1\}^2$ via $F(x,y) = (x, f(x) + y \pmod 2)$.

The problem was to determine whether $f(0) \stackrel{?}{=} f(1)$. Our solution was to compute F on a superposition of two states.

$$F\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|0\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|0\rangle\right) = \frac{1}{2}\left((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right) \otimes (|0\rangle - |1\rangle)$$

where we apply a rotation to the possible outcomes. Our key transformation is

$$(x,y) \mapsto \left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y\right)$$

which is a *rotation by* $45°$.

We discuss a generalization of this which is the Discrete Fourier transformation.

Definition 22.1 (Discrete Fourier Transform)

Given $x_1, x_2, \ldots, x_N$, the Discrete Fourier Transform (DFT) is a new sequence $y_1, y_2, \ldots, y_N$ defined via:

$$y_k = \frac{1}{\sqrt{N}} \sum_j e^{-\frac{2\pi i}{N} \cdot jk} x_j$$

Note that this is the discrete analog of the ordinary Fourier Transform:

$$f(x) \rightsquigarrow \hat{f}(y) = \int e^{2\pi i x y} f(x)\, dx$$

Example 22.2

For $N = 2$, we have x_1, x_2 so

$$y_1 = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2$$
$$y_2 = \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2$$

we note that these are efficiently computable in the quantum setting.

§22.2 Shor's Algorithm

Question. Given a black box that implements a function $f : \mathbb{Z} \to X$ (assumed to be periodic). What is the period of f?

This is to say, we're promised that $f(x) = f(x + n)$ for some n, we are tasked to find n.

Classically, we take

$$f(1), f(2), \ldots, f(n + 1)$$

where $f(n + 1) = f(1)$. The runtime is $\mathcal{O}(n)$.

We try to solve this using a quantum algorithm:

1. Choose N large power of 2 (we want $N = \mathcal{O}(n^2)$).
2. Prepare the state
$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle \otimes |f(j)\rangle .$$
3. Apply DFT to first register. That is,
$$\frac{1}{N} \sum_{k,x} \left(\sum_{j:f(j)=x} e^{-\frac{2\pi i}{N} jk} \right) |k\rangle \oplus |x\rangle$$
4. Measure first register.

How big is this sum?

$$\sum_{j:f(j)=x} e^{-\frac{2\pi i}{N} jk}$$

We note that this sum

$$\sum_{j:f(j)=x} e^{-\frac{2\pi i}{N} jk} \approx \sum_{j:f(j)=x} e^{-\frac{2\pi i}{N}(j+n)k} = e^{-\frac{2\pi ink}{N}} \cdot \sum_{j:f(j)=x} e^{-\frac{2\pi i}{N} jk}$$

therefore

$$\sum_{j:f(j)=x} e^{-\frac{2\pi i}{N}jk} \approx 0 \quad \text{unless} \quad e^{-\frac{2\pi ink}{N}} \approx 1 \iff nk \text{ is close to a multiple of } N$$

This is to say, k is approximately a multiple of $\frac{N}{n}$. So we get $\frac{N}{n}$ out of this register.

Why might I get a multiple?

Example 22.3

Consider sequence

$$\boxed{0,1,0,1,0,1,0,0},0,1,0,1,0,1,0,0,,0,1,0,1,0,1,0,0,\ldots$$

where the period is 8. We'll usually get $\approx \frac{N}{\text{period}}$ unless there is a smaller "almost-period" where we might get $\frac{N}{\text{almost-period}}$ but the almost-period divides the period.

The runtime of this algorithm is $\approx \log n$.

§22.3 Breaking Encryption

§22.3.1 Integer Factorization

We're given $N = pq$. We pick $x \in \mathbb{Z}/N\mathbb{Z}$ and consider the function $f(j) = x^j \pmod N$. Applying Shor's Algorithm, we can determine the period of f, which is the order of x, which is a factor of $(p-1)(q-1)$. More precisely, $\frac{(p-1)(q-1)}{\text{something small}}$. We recover $(p-1)(q-1)$ and we can factor N easily. This is bad news for RSA!

§22.3.2 Quantum Elgamal/DLP

We have some group G and $g \in G$ where we want to recover k from $x = g^k$.

Consider function $f(a,b) = x^a \cdot g^{-b}$ (we do a 2-dimensional DFT). $f(a,b) = f(a+1, b+k)$. So $(1,k)$ is the period of f. This solves the discrete log problem *in any group*!

§23 April 6, 2022

§23.1 Lattices and Cryptography

The advantage of lattice based cryptography is that they are quantum resistant.

We'll use a *toy example* as a warm-up[12].

Example 23.1

Suppose we have some q that is public knowledge (any integer).

Alice will choose reasonably small numbers f, g

$$0 < f < \sqrt{\frac{q}{2}}$$
$$\sqrt{\frac{q}{4}} < g < \sqrt{\frac{q}{2}}$$

which will constitute her private keys. She'll compute $h = f^{-1} \cdot g \pmod{q}$ which will be her public key. We'll assume that f, g, q are all pairwise relatively prime.

Bob, encrypting message m, satisfying $0 < m < \sqrt{\frac{q}{4}}$:

1. Choose random r with $0 < r < \sqrt{\frac{q}{2}}$.
2. Compute ciphertext $c = r \cdot h + m \pmod{q}$ to send to Alice.

Alice, to decrypt the message, will do the following:

1. Calculate $a \equiv f \cdot c \pmod{q}$.
2. Calculate $b \equiv f^{-1}a \pmod{g}$.

Why does this work?

$$\begin{aligned} a \equiv f \cdot c \equiv f(r \cdot h + m) &\equiv f(r \cdot f^{-1} \cdot g + m) \\ &\equiv r \cdot g + f \cdot m \end{aligned}$$

and we rely on the fact that $r \cdot g + f \cdot m < q$ since

$$0 < rg + fm < \sqrt{\frac{q}{2}}\sqrt{\frac{q}{2}} + \sqrt{\frac{q}{2}}\sqrt{\frac{q}{4}} \leq q$$

Thus $a \equiv rg + fm$ (exactly!). Then $b \equiv f^{-1}a \equiv f^{-1}(rg + fm) \pmod{g} \equiv m \pmod{g}$. $m < \sqrt{\frac{q}{4}} < g$ thus $b = m$ exactly.

We can implement this in code (again):

```
1 from gcd import *
2
```

[12] This isn't even secure classically.

```
q = 320984712309487123509238471251

while True:
    f = randrange(int(sqrt(q/2)))
    g = randrange(int(sqrt(q/4))+ 1, int(sqrt(q/2)))
    if gcd(f, g) == 1 and gcd(f, q) == 1:
        break

h = (ext_gcd(f, q)[0] * g) % q

def e(m):
    r = randrange(sqrt(q/2))
    c = (r * h + m) % q
    return c

def d(c):
    a = (f * c) % q
    b = (ext_gcd(f, g)[0] * a) % g
    return b
```

Question. What does Eve need to do?

Eve knows q, h and wants to figure out f, g with $f \cdot h \equiv g \pmod{q}$ and f, g small ($\mathcal{O}(\sqrt{q})$).

We write this as a vector equation:

$$f \cdot \begin{pmatrix} 1 \\ h \end{pmatrix} - r \cdot \underbrace{\begin{pmatrix} 0 \\ q \end{pmatrix}}_{V_1} = \underbrace{\begin{pmatrix} f \\ g \end{pmatrix}}_{V_2}$$

where f, r are unknown integers and $\begin{pmatrix} f \\ g \end{pmatrix}$ is an unknown vector. The known vectors are on the left.

Our goal is to find a short vector in

$$\{a_1 v_a + a_2 v_2 \mid a_1, a_2 \in \mathbb{Z}\}$$

Definition 23.2 (Lattice)

Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ be linearly independent vectors vectors (so $n \leq m$).

The lattice generated by $v_1, \ldots, v_n$ is:

$$\{a_1 v_1 + \cdots + a_n v_n : a_1, \ldots, a_n \in \mathbb{Z}\}$$

Remark 23.3. There is a fast algorithm to find *short vectors* in a 2D lattice (which is the one above). Then, Eve can use this to break the above cryptosystem. *(We will see this later.)*

§23.2 Subset Sum Cryptosystem

The subset sum problem is as follows: Given $M_1, M_2, \ldots, M_n \in \mathbb{Z}$: find a subset whose sum is S.

Example 23.4

We take $M = \{2, 3, 5, 8\}$ and $S = 10$. $S = 2 + 8 = 2 + 3 + 5$. (We note that the subset is not necessarily unique).

The idea is as follows:

Alice chooses $M_1, M_2, \ldots, M_n$. Bob has a message $x_1, x_2, \ldots, x_n$ where each $x_i \in \{0, 1\}$. Bob computes $\sum x_i M_i$ (that is, Bob's message specifies a subset of M_i's) and sends to Alice.

Alice has to recover which subset Bob sent her. Alice needs to have chosen $M_1, \ldots, M_n$ so that a) the solution to subset-sum is unique, b) it has some secret structure to solve subset sum...

Continued next time.

§24 April 8, 2022

§25 April 11, 2022

§25.1 Merkle-Hellman Public Key Cryptosystem

Alice picks a superincreasing sequence $\boxed{r_1, r_2, \ldots, r_n}$. Alice also picks $\boxed{A, B}$ relatively prime integers with $B > 2 \cdot r_n$. This is Alice's private key.

Alice computes sequence

$$\boxed{M_i = A \cdot r_i \mod B}$$

which is her public key.

Bob encrypts message $(x_1, \ldots, x_n)$ by calculating

$$c = \sum x_i M_i$$

and sending it to Alice.

Alice decrypts c as follows.

$$c = \sum x_i M_i = \sum x_i A r_i \mod B$$
$$A^{-1} \cdot c = \sum x_i r_i \mod B$$

where $x_i \cdot r_i$ is less than B. So decryption is to calculate $c' = A^{-1} \cdot c \pmod{B}$. Then write

$$c' = \sum x_i \cdot r_i$$

using algorithm for superincreasing sequences.

We implement as follows:

```
from sssi import *

R = [1, 3, 9, 20, 50, 101]
A = 485
B = 1009
print(f"A and B coprime? {gcd(A, B) == 1}")

M = [i * A % B for i in R]
print(f"M = {M}")


def e(X):
    return sum([X[i] * M[i] for i in range(len(M))])


c = e([1, 1, 1, 0, 1, 1])

print(f"c = {c}")


def d(c):
    cp = ext_gcd(A, B)[0] * c % B
    return sssi(R, cp)


print(f"d(c) = {d(c)}")
```

§25.2 Merkle-Hellman & Lattices

We consider the lattice L generated by

$$V_1 = (2, 0, 0, \ldots, 0, M_1)$$
$$V_2 = (0, 2, 0, \ldots, 0, M_2)$$
$$\vdots$$
$$V_n = (0, 0, 0, \ldots, 2, M_n)$$
$$V_{n+1} = (1, 1, 1, \ldots, 1, c)$$

where c is our ciphertext. We note that L contains

$$\left(\sum_{i=1}^{n} x_i v_i\right) - v_{n+1}$$

$$(2x_1 - 1, 2x_2 - 1, \ldots, 2x_n - 1, 0) = (\pm 1, \pm 1, \ldots, \pm 1, 0)$$

This vector is *short*! It has length about $\sqrt{n}$. The upshot is that we can reduce breaking M-H to finding short vectors in lattices.

§25.3 Vector Spaces and Inner Products: *Review*

Let $V \subseteq \mathbb{R}^m$ be a subspace of dimension n. We must have $n \leq m$. This subspace has a basis

$$(v_1, v_2, \ldots, v_n)\,.$$

Any vector $w \in V$ can be written as $w = a_1 v_1 + \cdots a_n v_n$ uniquely. Let's say we have

$$\begin{aligned} w_1 &= a_{11} v_1 + \cdots a_{1n} v_n \\ w_2 &= a_{21} v_1 + \cdots a_{2n} v_n \\ &\vdots \\ w_n &= a_{n1} v_1 + \cdots a_{nn} v_n \end{aligned}$$

When is $\{w_i\}$ a basis? It is when we can also express

$$\begin{aligned} v_1 &= a_{11} w_1 + \cdots a_{1n} w_n \\ v_2 &= a_{21} w_1 + \cdots a_{2n} w_n \\ &\vdots \\ v_n &= a_{n1} w_1 + \cdots a_{nn} w_n \end{aligned}$$

That is, we can define a change-of-basis matrices

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{nn} \end{pmatrix}$$

where $A \cdot B = I$. We also have that $(\det A)(\det B) = 1$ so $\det A \neq 0$. Conversely, Cramer's rule gives a formula for $A^{-1} = \frac{1}{\det A}(\cdots)$ that involves division by $\det A$.

So we have that $\{w_i\}$ is a basis $\iff \det A \neq 0$.

Question. What if instead of talking about subspaces of a vector space, we talked about a lattice?

We restrict $a_i \in \mathbb{Z}$. When is $\{w_i\}$ a basis for the same lattice? The logic is the same as saying $A \cdot B = I$ for some integer matrix B. This certainly implies $\det A = \det B = 1$. Then $\det A = \pm 1$.

For lattices: $\{w_i\}$ is a basis if and only if $\det A = \pm 1$.

§26 April 15, 2022

§26.1 Vector Spaces and Inner Products

For

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$$

we can compute $||v|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

For two vectors

$$\mathbf{v} = (v_1, v_2, \ldots, v_n)$$
$$\mathbf{w} = (w_1, w_2, \ldots, w_n)$$

both in $\mathbb{R}^n$, we have

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = ||\mathbf{v}|| \cdot ||\mathbf{w}|| \cos\theta$$

Definition 26.1

An orthogonal basis $\{\mathbf{v}_i\}$ is a basis with

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \qquad \text{for } i \neq j$$

Definition 26.2

An orthonormal basis is an orthogonal basis with $||\mathbf{v}_i|| = 1$, $\forall i$.

Let $\{\mathbf{v}_i\}$ be an orthonormal basis, and

$$\mathbf{u} = a_1 \mathbf{v}_1 + \cdots a_n \mathbf{v}_n$$
$$\mathbf{w} = b_1 \mathbf{v}_1 + \cdots b_n \mathbf{v}_n$$

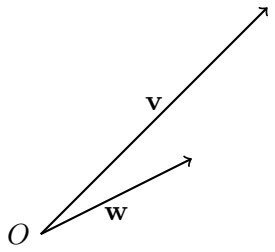


Figure 1: Not orthogonal basis.

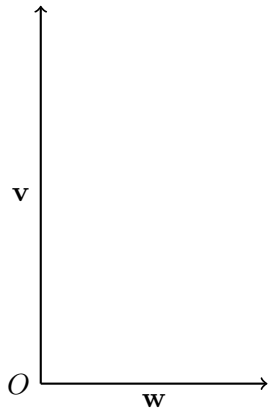


Figure 2: Orthogonal but not orthonormal

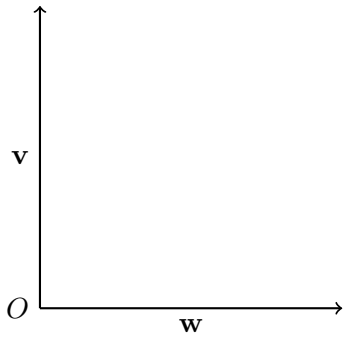


Figure 3: Orthonormal basis

then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{w} &= \left(\sum_i a_i\mathbf{v}_i\right)\cdot\left(\sum_j b_j\mathbf{v}_j\right)\\ &= \sum_{i,j} a_ib_j\mathbf{v}_i\mathbf{v}_j\\ &= \sum a_i\cdot b_j\end{aligned}$$

Let $V \subseteq \mathbb{R}^m$ be a subspace of dimension $n \leq m$ have basis $\{\mathbf{v}_1, \mathbf{v}_w, \ldots, \mathbf{v}_n\}$.

Example 26.3

$V \subseteq \mathbb{R}^3$, a 2 dimensional subspace spanned by

$$(1, 0, -1) \quad (1, -1, 0)$$

which is the plane $x + y + z = 0$.

Question. How do we find an orthogonal basis for V?

We use an algorithm called *Gram-Schmidt*.

We first fix $\mathbf{v}_1' = \mathbf{v}_1$. We then pick a $\mathbf{v_2}$ and compute $\mathbf{v}_2' = \mathbf{v}_2 - \mu_{21}\mathbf{v}_1$ where $\mu_{21}\mathbf{v}_1$ is the projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$. What should we select for μ_{21}?

We want $\mathbf{v}_1 \cdot (\mathbf{v}_2 - \mu_{21}\mathbf{v}_1) = 0$. This is to say

$$\mathbf{v}_1 \cdot \mathbf{v}_2 - \mu_{21}(\mathbf{v}_1\mathbf{v}_1) = 0$$
$$\mu_{21} = \frac{\mathbf{v}_1'\mathbf{v}_2}{\mathbf{v}_1' \cdot \mathbf{v}_1'}$$

Let's say we get a $\mathbf{v}_3$. We have $\mathbf{v}_3' = \mathbf{v}_3 - \mu_{31}\mathbf{v}_1' - \mu_{32}\mathbf{v}_2'$.

We now want

$$\mathbf{v}_1'(\mathbf{v}_3 - \mu_{31}\mathbf{v}_1' - \mu_{32}\mathbf{v}_2') = 0 \qquad \mathbf{v}_2'(\mathbf{v}_3 - \mu_{31}\mathbf{v}_1' - \mu_{32}\mathbf{v}_2') = 0$$
$$\mathbf{v}_1' \cdot \mathbf{v}_3 - \mu_{31}(\mathbf{v}_1' \cdot \mathbf{v}_1') = 0 \qquad \mathbf{v}_2' \cdot \mathbf{v}_3 - \mu_{32}(\mathbf{v}_2' \cdot \mathbf{v}_2') = 0$$
$$\mu_{31} = \frac{\mathbf{v}_1' \cdot \mathbf{v}_3}{\mathbf{v}_1' \cdot \mathbf{v}_1'} \qquad \mu_{32} = \frac{\mathbf{v}_2' \cdot \mathbf{v}_3}{\mathbf{v}_2' \cdot \mathbf{v}_2'}$$

So in general, we want

$$\mathbf{v}_j' = \mathbf{v}_j - \sum_{i<j} \frac{\mathbf{v}_i' \cdot \mathbf{v}_j}{\mathbf{v}_i' \cdot \mathbf{v}_i'} \cdot \mathbf{v}_i'$$

In code, we can implement this as follows (not using numpy):

```
from sympy import sqrt, Rational


def dot(v: list[int], w: list[int]) -> int:
    """
    Computes the dot product of two vectors.

    >>> dot([1, 2], [3, 4])
    11
```

```
10
11      >>> dot([1, 2, 1], [3, 4, 3])
12      14
13
14      param v: a vector
15      param w: a vector
16      return: the dot product of v and w
17      """
18      return sum(v_i * w_i for v_i, w_i in zip(v, w))
19
20
21  def length(v: list[int]) -> float:
22      """
23      Computes the length of a vector.
24
25      >>> length([3, 4])
26      5.0
27
28      >>> length([1, 2, 3])
29      sqrt(14)
30
31      param v: a vector
32      return: the length of v
33      """
34      return sqrt(dot(v, v))
35
36
37  def gs(L: list[list[int]]) -> list[list[int]]:
38      """
39      Computes an orthogonal basis of span(L).
40
41      >>> gs([[1, 0, -1], [1, -1, 0]])
42      [[1.0, 0, -1.0], [0.5, 1.0, 0.5]]
43
44      param L: a list of basis vectors
45      return: an orthogonal basis using the Gram-Schmidt process
46      """
47      M = []
48      for v in L:
49          for w in M:
50              mu = Rational(dot(w, v), dot(w, w))
51              v = [v_i - mu * w_i for v_i, w_i in zip(v, w)]
52          M.append(v)
53      # Uncomment below if we want orthonormal basis:
54      # M = [[v_i / length(v) for v_i in v] for v in M]
55      return M
```

What about with lattices? *** (pretty diagram)

Any subgroup of a lattice (in particular, of $\mathbb{Z}^n$) is itself a lattice.

Definition 26.4
A subgroup of $\mathbb{Z}^n$ is called an integral lattice.

Question. Is any subgroup of $\mathbb{R}^n$ a lattice?

Clearly not! $\mathbb{R}^n$ itself is not a lattice. $\mathbb{Q}^n$ is similarly not a lattice.

We want to characterize subgroups of $\mathbb{R}^n$ that are lattices.

Definition 26.5
A subset $S \subseteq \mathbb{R}^n$ is discrete if for any $x \in S$: there is some $\epsilon > 0$ such that

$$\{y \in S \mid ||x - y|| < \epsilon\} = \{x\}$$

***another pretty picture

Theorem 26.6
A subgroup of $\mathbb{R}^n$ is a lattice iff it is discrete.

Proof. HW problem. □

§27 April 15, 2022

§28 April 18, 2022

§28.1 Midterm 2 Review

- Apologies for poor communication.
- Feedback is welcome—will send out form.

Average: 28/40.

"A-level work" would equate to doing around 3 problems,
"B-level work" would equate to doing around 2 problems, and
"C-level work" would equate to doing around 1 problem.

Problem 28.1. *About 3/4 solved.*

Idea: exploit something special about n being a Carmichael number. That is that $a^{N-1} \equiv 1 \pmod{N}$. We can do something inspired by the Miller-Rabin test. We then have

$$\left(a^{\frac{N-1}{2}} \equiv 1 \pmod{N}\right)$$

If $a^{\frac{N-1}{2}} \equiv 1$, then we decrement the exponent by a factor of two. We then check

$$\left(a^{\frac{N-1}{4}} \equiv 1 \pmod{N}\right)$$

and so on. Eventually, we'll find a nontrivial square root of 1. This produces a factorization, since

$$\begin{aligned} x^2 &\equiv 1 \pmod{n} \\ (x-1)(x+1) &\equiv 0 \pmod{n} \end{aligned}$$

so $\gcd(x \pm 1, n)$ likely allows us to recover a factor of N. We run through this multiple times with different values of a.

Problem 28.2. *Hardest problem, 1/2 solved.*

Idea: Implement some reasonably general-purpose factorization method:

1. Lenstra's Elliptic Curve Factorization.
2. Quadratic Sieve.
3. Pollard ρ method.

Things that would not work:

1. Trial division.
2. Pollard $p-1$.

The factorization was 15×35 digits, which gives a relatively equal runtime for Elliptic Curve and Quadratic Sieve.

Problem 28.3. *About 2/3 solved (generally speaking, lost 3 points on it).*

Idea: solve DLP. Babystep-Giantstep, as in class, yields 7/10 due to excessive memory usage. 4 people solved this problem in a way that didn't use a shit-ton of RAM, with 3 distinct solutions.

How to solve DLP without using tons of RAM:

1. Babystep-Giantstep with fewer babysteps (B babysteps). We need $B + \frac{N}{B}$ time to solve this, which is minimized if $B \equiv N^{0.5}$. We take a smaller $B \equiv N^{0.3}$ or something, and our time would be $N^{0.7}$, still within the bounds.

2. Index calculus. Time/Memory are asymptotically small compared to Babystep-Giantstep.

3. Pollard ρ method *(discussion of which is in textbook)*. Gives runtime of $N^{0.5}$ with minimal memory usage.

Problem 28.4. *This was a fairly easy problem.*

§29 April 20, 2022

§29.1 Short Vectors

Question. How short is the shortest vector in a lattice L?

A more general question we could ask:

Question. When does some region contain a nontrivial lattice point?

Theorem 29.1 (Minkowski's Theorem)
Let $L \subseteq \mathbb{R}^n$ be a lattice of dimension n. Let $S \subseteq \mathbb{R}^n$ be a *bounded symmetric convex* set.

If $\mathsf{Vol}(S) > 2^n \det(L)$, then $S \cap L$ contains a nonzero lattice point.

Definition 29.2 (Bounded Set)
$\{$Lengths of vectors in $S\}$ is bounded.

In other words, there is some ball that contains S.

Definition 29.3 (Symmetric Set)
If $\mathbf{v} \in S$, then $-\mathbf{v} \in S$.

Definition 29.4 (Convex Set)
If $\mathbf{v}, \mathbf{w}$, then the line segment connecting $\mathbf{v}$ and $\mathbf{w}$ is a subset of S.

Proof of Minkowski's Theorem. Let $\mathcal{F}$ be a fundamental domain.

Any vector $\mathbf{w} = t(\mathbf{w}) + v(\mathbf{w})$ where $t(\mathbf{w}) \in \mathcal{F}$ and $v(\mathbf{w}) \in L$.

Consider map $t : \frac{1}{2}S \mapsto F$ by sending every vector $t : \mathbf{w} \mapsto t(\mathbf{w})$.

What does t do to volume? We cut S up into finite number of regions, and 'cut-and-paste' them into the fundamental domain.

Locally, t preserves volume. When must two points in $\frac{1}{2}S$ be sent to the same point in F? When we have 'carpet' with area greater than room area. That is to say, $\mathsf{Vol}(\frac{1}{2}S) > \mathsf{Vol}(\mathcal{F})$ implies that there is an overlapping point.

This is to say

$$\begin{aligned}\mathsf{Vol}\left(\frac{1}{2}S\right) &> \mathsf{Vol}(\mathcal{F}) \\ \frac{1}{2^n}\mathsf{Vol}(S) &> \mathsf{Vol}(\mathcal{F}) = \det(L) \\ \mathsf{Vol}(S) &> 2^n \det(L)\end{aligned}$$

So given this inequality, there are two points in $\frac{1}{2}S$ such that $t\left(\frac{1}{2}\mathbf{w}_1\right) = t\left(\frac{1}{2}\mathbf{w}_2\right)$. Then we know that

$$\frac{1}{2}\mathbf{w}_1 - \frac{1}{2}\mathbf{w}_2 \in L$$

So then consider

$$\frac{1}{2}\mathbf{w}_1 - \frac{1}{2}\mathbf{w}_2 = \frac{1}{2}(\mathbf{w}_1 - \mathbf{w}_2)$$

which is the midpoint of $\mathbf{w}_1$ and $\mathbf{w}_2$, which is in S and in L. So S contains a nonzero lattice point. □

Theorem 29.5 (Variant of Minkowski's Theorem)

If $S \subseteq \mathbb{R}^n$ is bounded, symmetric, convex and *closed* set, then if

$$\mathsf{Vol}(S) \geq 2^n \det(L)$$

$S \cap L$ contains a nonzero lattice point.

Definition 29.6 (Closed Set)

Every limit point of S is contained in S.

We added the condition that S be closed, and changed our bound to be a $\geq$.

Proof of variant. For any k:

$$\left(1 + \frac{1}{k}\right) S \cap L$$

contains $\mathbf{v}_k \neq \mathbf{0} \in L$ (which is true by our first version).

The sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ is a sequence in $2S \cap L$. $2S$ is bounded, so we have a finite set of lattice points. There's some $\mathbf{v} \neq \mathbf{0}$ is contained in $\bigcap_k \left(1 + \frac{1}{k}\right) S = S$ because S is closed. □

Corollary 29.7 (Hermite's Theorem)
Let L be a lattice of dimension n in $\mathbb{R}^n$. Then, L contains a vector $\mathbf{v}$ with

$$||\mathbf{v}|| \leq \sqrt{n} \cdot \det(L)^{\frac{1}{n}}$$

Proof. Application of Minkowski's Theorem. Apply Minkowski's Theorem to

$$\left\{(x_1, \ldots, x_n) \;\middle|\; |x_i| \leq \det(L)^{1/n}\right\}$$

which is a cube with side length $2 \cdot \det(L)^{1/n}$. So $\mathsf{Vol}(S) = 2^n \cdot \det(L)$. The diagonal has length $\sqrt{n}\det(L)^{1/n}$. □

A variant of Hermite's Theorem is that we can find an entire basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that

$$||\mathbf{v}_1|| \cdot ||\mathbf{v}_2|| \cdots ||\mathbf{v}_n|| \leq n^{n/2} \det(L)$$

and we define the Hadamard ratio to be

$$\mathcal{H} = \left(\frac{\det(L)}{||\mathbf{v}_1|| \cdot ||\mathbf{v}_2|| \cdots ||\mathbf{v}_n||}\right)^{1/n}$$

where $0 < \mathcal{H} \leq 1$ and $\mathcal{H} = 1$ when our basis is orthogonal.

§30 April 22, 2022

§30.1 Short Vectors *continued*

Recall: last class we stated Hermite's Theorem:

Theorem (Hermite's Theorem)
Let L be a lattice of dimension n. Then $0 \neq v \in L$ with

$$||\mathbf{v}|| \leq \sqrt{n} \cdot \det(L)^{\frac{1}{n}}$$

Proof. Apply Minkowski's Theorem to the cube

$$S = \left\{(x_1, \ldots, x_n) \;\middle|\; -\det(L)^{1/n} \leq x_i \leq \det(L)^{1/n}\right\}$$

Minkowski's states that S contains a nonzero lattice vector. By inspection, this lattice vector $\mathbf{v} \in S$ implies that

$$||\mathbf{v}|| \leq \underbrace{\sqrt{(\det(L)^{1/n})^2 + \cdots + (\det(L)^{1/n})^2}}_{n \text{ times}} = \sqrt{n}(\det(L)^{1/n})$$

□

Theorem (Variant of Hermite's Theorem)

There is a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with

$$||\mathbf{v}_1|| \cdot ||\mathbf{v}_2|| \cdots ||\mathbf{v}_n|| \leq n^{n/2} \det(L)$$

Definition 30.1 (Hadamard Ratio)

The Hadamard Ratio is

$$\mathcal{H} = \left(\frac{|\det(L)|}{||\mathbf{v}_1|| \cdot ||\mathbf{v}_2|| \cdots ||\mathbf{v}_n||} \right)^{1/n}$$

The variant of Hermite's Theorem says that there is a basis for which $\mathcal{H} \geq \frac{1}{\sqrt{n}}$. For any basis, $0 < \mathcal{H} \leq 1$. $\mathcal{H} = 1$ if and only if our basis is orthogonal.

This ratio makes precise how orthogonal our basis is.

We can write this in code (*now enhanced with NumPy!*):

```
from numpy import *
from numpy.linalg import *


def hadamard_ratio(L):
    """
    Computes the Hadamard ratio of a list of vectors.
    """
    H = abs(det(array(L)))
    for v in L:
        H /= norm(v)
    return H ** (1 / len(L))
```

Instead of using a hypercube, we can use a hypersphere. Let

$$S = B_R(\mathbf{0}),$$

a ball with radius R centered at 0. Minkowski's theorem says S contains a nonzero lattice vector if $\text{Vol}(B_R(\mathbf{0})) \geq 2^n \cdot \det(L)$. Let $\text{Vol}(B_R(\mathbf{0})) = C_n \cdot R^n$ where C_n is the volume of a unit ball in n dimensions.

This is also to say that

$$R \geq \frac{2}{C_n^{1/n}} \det(L)^{1/n}$$

Fact from analysis/calculus (Stirling's Formula):

$$\lim_{n\to\infty} C_n^{1/n} \cdot \sqrt{n} = \sqrt{2\pi e}.$$

The key point is

$$R \geq \frac{2}{C_n^{1/n}} \det(L)^{1/n} \approx \sqrt{\frac{2}{\pi e}} \sqrt{n} (\det(L))^{1/n}$$

Question. What is the real truth? Given a random lattice, how long is the shortest vector? What is the actual length (probably)?

How many lattice points *"should"* a ball of radius R contain? Probably the volume divided by the volume of the fundamental domain:

$$\frac{\mathsf{Vol}(S)}{\mathsf{Vol}(\mathcal{F})} = \frac{\mathsf{Vol}(S)}{\det(L)}.$$

So when *should* $B_R(\mathbf{0})$ contain nonzero $\mathbf{v} \in L$? This is probably when $\mathsf{Vol}(B_R(\mathbf{0})) \geq \det(L)$.

It's around when

$$R \geq \sqrt{\frac{1}{2\pi e}} \sqrt{n} (\det(L))^{1/n}$$

Definition 30.2 (Gaussian Expected Shortest Length)
$\sigma(L) = \sqrt{\frac{n}{2\pi e}} \cdot (\det(L))^{1/n}$ is called the Gaussian expected shortest length.

We *expect* $||\mathbf{v}_{\mathsf{shortest}}|| \approx \sigma(L)$. We *know* $||\mathbf{v}_{\mathsf{shortest}}|| \lesssim 2 \cdot \sigma(L)$

§30.2 Babai's Algorithm

Goal: we want to solve, approximately, the closest vector problem. We have $L \subseteq \mathbb{R}^n$ with basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$. We have some $\mathbf{w} \in \mathbb{R}^n$. We want to find a close lattice vector to $\mathbf{w}$. That is, we want

$$a_1, a_2, \ldots, a_n \in \mathbb{Z}$$

where $||\mathbf{w} - \sum a_i \mathbf{v}_i||$ is small.

Method:

1. Write $\mathbf{w} = \alpha_1 \mathbf{v}_1 + \cdots \alpha_n \mathbf{v}_n$.

2. Round α_i to nearest integer a_i.

Question. When does this work well?

We implement this in code:

```
from numpy import *
from numpy.linalg import *


def babai(L, w):
    """
    param L: lattice
    param w: vector
    """
    M = array(L).T
    W = array([w]).T
    a = vectorize(round)(inv(M) @ W)
    wp = M @ a
    return ([i[0] for i in a], [i[0] for i in wp])
```

§31 April 25, 2022

§31.1 Babai's Algorithm *continued*

Recall: We revisit some topics from before. We reverse our original course and use sage instead of numpy. (For symbolic math).

```
def hadamard_ratio(L):
    H = abs(L.det())
    for v in L:
        H /= v.norm()
    return ((H ** (1 / L.dimensions()[0]))).n()

L = random_matrix(ZZ, 3, x=-10, y=10)

def babai(L, w):
    a = w * L.inverse()
    a = a.apply_map(lambda x : x.round())
    return (a, a * L)

w = random_vector(ZZ, 3, x=-100, y=100)

print(f"Hadamard Ratio: {hadamard_ratio(L)}")
print(f"Babai: {babai(L, w)}")
```

Proposition 31.1

If $\mathbf{v}_i$ is orthogonal, then Babai's Algorithm solves CVP.

Proof. We have some

$$\mathbf{w} = \sum a_i \mathbf{v}_i$$

and we want to minimize

$$\begin{aligned} \left\| \mathbf{w} - \sum \alpha_i \mathbf{v}_i \right\|^2 &= \left\| \sum (a_i - \alpha_i) \mathbf{v}_i \right\|^2 \\ &= \sum (a_i - \alpha_i)^2 ||\mathbf{v}_i||^2 \end{aligned}$$

and this is minimized when α_i is the closest integer to a_i for all i. □

We start using this theory of lattices to develop cryptosystems...

§31.2 GGH Cryptosystem: Digital Signatures

Alice: Chooses $\{\mathbf{v}_i\}$ a good basis for some lattice. This is Alice's *private key.*

Alice computes a *bad basis* $\mathbf{w}_i = U \cdot \mathbf{v}_i$ (for $n \times n$ matrix U with $\det U = 1$). Note that $\mathbf{w}_i$ is still a basis for the same lattice.

How does Alice sign a document $\mathbf{d} \in \mathbb{Z}^n$*?*

1. Apply Babai's algorithm to find a close lattice vector.
2. Express answer $\sum a_i \mathbf{w}_i$ in the bad basis.

How does Bob verify that this is the correct document?

Bob checks how close the document $\mathbf{b}$ to $\sum a_i \mathbf{w}_i$.

How do we generate U*?*

1. Start with identity matrix I.
2. Apply some random row operations.

(Code is in lattices.ipynb)

§32 April 27, 2022

§32.1 GGH Cryptosystem: Public-Key Cryptosystem

Recall: last time we talked about GGH digital signatures.

We're going to use similar ideas to construct a public-key cryptosystem. As before, we'll have the same public/private key setup as last time.

Alice: will choose a random basis $\{\mathbf{v}_i\}$ which will be her private key, which is a *sufficiently good* basis. Alice will then compute $\{\mathbf{w}_i\}$ which is a bad basis (by multiplying by some $U \in \mathsf{SL}_n(\mathbb{Z})$) which is her public key.

Bob: will encrypt message $\{m_i\}$ by computing

$$\mathbf{c} = \sum m_i \mathbf{w}_i + \mathbf{r}$$

where $\mathbf{r}$ is a randomly generated short vector.

Alice: Find the closest lattice point to $\mathbf{c}$ using Babai's algorithm, and express it in the public basis $\{\mathbf{w}_i\}$.

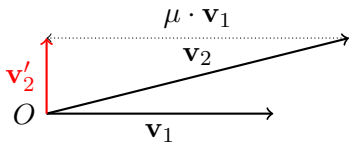
Code in lattices.ipynb.

§32.2 Lattice Reduction

Warm-up: Gaussian Lattice Reduction.

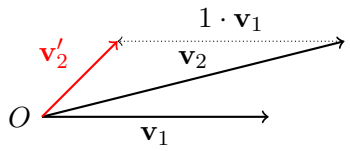
We'll see how to find the shortest vector in a 2-dimensional lattice.

With Gram-Schmidt, we made an orthgonal basis.



Using Gram-Schmidt, we have $\mathbf{v}_2' = \mathbf{v}_2 - \mu \cdot \mathbf{v}_1$ where $\mu = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}$.

With lattice reduction, we round μ to subtract by an integer multiple of $\mathbf{v}_2$ instead.



Lattice reduction gives us

$$\mathbf{v}_2' = \mathbf{v}_2 - \lfloor \mu \rceil \cdot \mathbf{v}_1$$

We do this again and again.

Code in lattices.ipynb.

We should prove that this gives us a reasonable basis when this algorithm finishes.

Proposition 32.1
This algorithm terminates.

Proof. Integer vectors $||\mathbf{v}_1||$ and $||\mathbf{v}_2||$ always decrease, and there are only *finitely many* lattice vectors that strictly decrease. □

Proposition 32.2
When this algorithm terminates, $\mathbf{v}_1$ is the shortest vector in the lattice.

Proof. At the end, we know that $||\mathbf{v}_2|| \geq ||\mathbf{v}_1||$ and that

$$-\frac{1}{2} \leq \mu = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \leq \frac{1}{2}.$$

Any vector $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$. So

$$\begin{aligned} ||\mathbf{w}||^2 &= a_1^2 \cdot ||\mathbf{v}_1||^2 + 2a_1a_2(\mathbf{v}_1 \cdot \mathbf{v}_2) + a_2^2||\mathbf{v}_2||^2 \\ &\geq a_1^2 \cdot ||\mathbf{v}_1||^2 + a_1a_2||\mathbf{v}_1||^2 + a_2^2||\mathbf{v}_2||^2 \\ &= (a_1^2 + a_1a_2 + a_2^2)||\mathbf{v}_1||^2 \end{aligned}$$

And $a_1^2 + a_1a_2 + a_2^2 = \frac{3}{4}(a_1 - a_2)^2 + \frac{1}{4}(a_1 + a_2)^2 > 0$. OTOH $a_1, a_2 \in \mathbb{Z}$ so $a_1^2 + a_1a_2 + a_2^2 \geq 1$.

So any $||\mathbf{w}||^2 \geq ||\mathbf{v}_1||^2$. □

§33 May 2, 2022

§33.1 LLL Reduction

Problem: Given a lattice, find a good basis for the given lattice. Let's say we have

$$\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$$
$$\mathcal{B}' = (\mathbf{v}_1', \mathbf{v}_2', \ldots, \mathbf{v}_n')$$

where $\mathcal{B}'$ is the corresponding vector space obtained from Gram-Schmidt. Note that this is *not* a basis of the lattice.

We have

$$\mathbf{v}_i' = \mathbf{v}_i - \sum_{j<i} \mu_{ij} \mathbf{v}_j'$$

where $\mu_{ij} = \frac{\mathbf{v}_i \cdot \mathbf{v}_j'}{\mathbf{v}_i' \cdot \mathbf{v}_i'}$.

What do we know about $\det\{\mathbf{v}_i\}$ and $\det\{\mathbf{v}_i'\}$? We note that $\det\{\mathbf{v}_i\} = \det\{\mathbf{v}_i'\}$. This is as our row/column operations (except scaling) don't change determinant.

Ideas:

We want some $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are approximately orthogonal.

We want $\mathbf{v}_1 \leq \mathbf{v}_2 \leq \cdots \leq \mathbf{v}_n$ (approximately sorted).

Definition 33.1 (LLL Reduction)

$\mathcal{B}$ is LLL reduced if

1. *We want a measure of orthogonality, so we bound μ_{ij} from Gram-Schmidt to a certain value.*

   We have that

   $$\boxed{|\mu_{ij}| = \left|\frac{\mathbf{v}_i \cdot \mathbf{v}_j'}{\mathbf{v}_i' \cdot \mathbf{v}_i'}\right| \leq \frac{1}{2}}$$

2. *We want our measure in a way that failure of exact orthogonality doesn't contribute.*

   We take the projection of $\mathbf{v}_{i-1}$ onto

   $$\langle \mathbf{v}_1, \ldots, \mathbf{v}_{i-2} \rangle^\perp$$

   and compare that with the length of the projection of $\mathbf{v}_i$ onto

   $$\langle \mathbf{v}_1, \ldots, \mathbf{v}_{i-1} \rangle^\perp$$

   We want

   $$\frac{3}{4} \left|\left| \text{ Projection of } \mathbf{v}_{i-1} \text{ onto } \langle \mathbf{v}_1, \ldots, \mathbf{v}_{i-2} \rangle^\perp \right|\right| \leq \left|\left| \text{ Projection of } \mathbf{v}_i \text{ onto } \langle \mathbf{v}_1, \ldots, \mathbf{v}_{i-1} \rangle^\perp \right|\right|$$

   Noting that $\frac{3}{4}$ was arbitrarly chosen as a 'fudge factor'.

   So we have

   $$\frac{3}{4}||\mathbf{v}_{i-1}||^2 \leq ||\mathbf{v}_i' + \mu_{i-1,i} \cdot \mathbf{v}_{i-1}'||^2 = ||\mathbf{v}_i||^2 + \mu_{i-1,i}^2 \cdot ||\mathbf{v}_{i-1}'||$$

   so then

   $$\boxed{||\mathbf{v}_i^2|| \geq \left(\frac{3}{4} - \mu_{i-1,i}^2\right) \cdot ||\mathbf{v}_{i-1}'||^2}$$

Our goal is that we want an algorithm to find an LLL basis. Additionally, once we do find an LLL basis, we want to know that it will be sufficiently orthogonal.

Code in lll.ipynb.