Math 1010: Analysis *Lecture Notes*

J. Serrano

Spring 2023

These are lecture notes for "Math 1010: Analysis: Functions of One Variable" taught at Brown University by Javi Gómez Serrano in the Spring of 2023.

These notes are taken by Jiahua Chen with gracious help and input from classmates. Please direct any mistakes/errata to me via email, or feel free to pull request the notes repository (`https://github.com/jchen/math1010-notes`).

Notes last updated March 21, 2023.

Contents

References

§1 March 14, 2023

By default, the quiz will be next Tuesday.

§1.1 Continuity, *continued*

Theorem 1.1

$f : X \to Y$ is continuous if and only if $\forall O \subset Y$ open, the preimage $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ is open.

Proof. We proved the right implication last time. We'll work on the left implication.

Suppose $\forall O \in Y, f^{-1}(O)$. Let $x_0 \in X, \epsilon > 0$. Then $B_\epsilon(f(x_0))$ is open in Y.

Then $f^{-1}(B_\epsilon(f(x_0)))$ is open in X. Since $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$, $\exists \delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$.

This implies that if $d(x, x_0) < \delta$, then $x \in f^{-1}(B_\epsilon(f(x_0)))$, then $d_Y(f(x), f(x_0)) < \epsilon$. Then f is continuous at x_0. $\square$

Corollary 1.2

$f : X \to Y$ is continuous in X if and only if for all $C \subset Y$ closed, $f^{-1}(C)$ is closed.

Proof. If C is closed, C^C is open, then $f^{-1}(C^C)$ is open, then $f^{-1}(C^C)^C$ is closed.

We claim $f^{-1}(C) = f^{-1}(C^C)^C$. If $x \in f^{-1}(C)$ iff $f(x) \in C$ iff $x \notin f^{-1}(C^C)$ iff $x \in f^{-1}(C^C)^C$.

This gives a biconditional between the second condition in theorem 1.1 and corollary 1.2. $\square$

Proposition 1.3

Let X be a metric space. Let $f, g : X \to \mathbb{R}$ be continuous at x_0. If $\alpha \in \mathbb{R}$, the following are continuous at x_0:

(1) $f + g$, fg, αf.

(2) f/g as long as $g(x_0) \neq 0$.

We'll state this without proof.

Theorem 1.4

Let X, Y, Z be metric spaces. Let $f : X \to Y$ and $g : Y \to Z$ with f continuous at x_0 and g continuous at $f(x_0)$. Then $g \circ f$ is continuous at x_0.

Proof. Let $\epsilon > 0$. Since g is continuous, $\exists \delta' > 0$ such that if indeed $d_Y(y, d(x_0)) < \delta'$, then $d_Z(g(y), (g \circ f)(x_0)) < \epsilon$.

Then by continuity of f, $\exists \delta > 0$ such that if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \delta'$ then $d_Z((g \circ f)(x), (g \circ f)(x_0)) < \epsilon$. Therefore $g \circ f$ is continuous. $\square$

Theorem 1.5

If $f : X \to Y$ is continuous, $E \subset X$ compact, then $f(E) = \{f(x) \mid x \in E\}$ is compact.

Proof. Let $\mathcal{C}$ bean open cover of $f(E)$. Consider $\mathcal{C}' = \{f^{-1}(O) \mid O \in \mathcal{C}\}$.

Every element of $\mathcal{C}'$ is open becuase f is continous. $\mathcal{C}'$ covers E since all elements of $f(E)$ are covered by $\mathcal{C}$. So $\mathcal{C}'$ is an open cover of E.

Then there exists a finite subcover $\{f^{-1}(O_1), \ldots, f^{-1}(O_n)\}$ of E. We claim that $\{O_1, \ldots, O_n\}$ is a finite subcover of $f(E)$. Let $y \in f(E)$ so $y = f(x)$ for some $x \in E$. $\exists O_i$ such that $x \in f^{-1}(O_i)$. Then $y = f(x) \in O_i$.

So this is a finite subcover of E. $\square$

Corollary 1.6

Let $f : X \to Y$ which is continuous, and $E \subset X$ compact, then f is bounded on E.

Proof. We know that $f(E)$ is compact by above, and if a set is compact it is bounded. $\square$

Corollary 1.7 (Extreme Value Theorem)

Let $f : X \to \mathbb{R}$ be continuous, and some $E \subset X$ compact.

Then f attains a maximum on E. That is, $\exists x_0 \in E$ such that $f(x_0) \geq f(x)$ $\forall x \in E$. This statement also applies to the minimum.

Proof. $f(E)$ is closed and bounded, so f contains its supremum, say $y = \sup(f(E))$. Since $y \in f(E)$, $\exists x$ such that $f(x) = y$. That is such an x_0. $\square$

Definition 1.8 (Uniform Continuity)

We say that $f : X \to Y$ is uniformly continuous if given $\epsilon > 0$, $\exists \delta > 0$ such that if x, y satisfy

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon.$$

We note that δ does not depend on x, y (normal continuity has δ depend on x, y).

Theorem 1.9

If $f : X \to Y$ continuous, and X is compact, then f is uniformly contunuous.

§2 March 16, 2023

Midterm will be the Thursday after Spring Break, quiz next Tuesday. Homework will be posted today, hopefully[1]

§2.1 Uniform Continuity

Theorem 2.1

If $f : X \to Y$ is continuous, and X is compact, then f is uniformly continuous.

The intuition is that we'll use the compactness of X to take some open subcover, taking the minimum of the finite subcover.

Proof. Let $\epsilon > 0$. For each $x \in X$, by continuity, $\exists \delta_x > 0$ such that if $d_X(x, x') < \delta_x$, then $d_Y(f(x), f(y)) < \frac{\epsilon}{2}$.

Consider the collection

$$\mathcal{C} = \left\{ B_{\frac{\delta_x}{2}}(x) \;\middle|\; x \in X \right\}$$

which is an open cover of X.

By compactness, exists a finite subcover

$$\left\{ B_{\frac{\delta_{x_1}}{2}}(x_1), \ldots, B_{\frac{\delta_{x_n}}{2}}(x_n) \right\}$$

Let $\delta = \min \left\{ \frac{\delta_{x_1}}{2}, \ldots, \frac{\delta_{x_n}}{2} \right\}$. We claim that if $d_X(x, y) < \delta$, then $\delta_Y(f(x), f(y)) < \epsilon$.

For each x, $\exists i$ such that $d_X(x, x_i) < \frac{\delta_{x_i}}{2}$. Then

$$\begin{aligned} d_X(x_i, y) &\le d_X(x, y) + d_X(x, x_i) \\ &< \delta + \frac{\delta_{x_i}}{2} \le \delta_{x_i} \end{aligned}$$

But then

$$\begin{aligned} d_Y(f(x), f(y)) &\le d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

which is as desired. $\square$

[1] Or the graders will, verbatim, 'be strangled'.

Definition 2.2 (Connectedness)

A metric space X is disconnected if $\exists O_1, O_2$, non-empty open sets such that $X = O_1 \cup O_2$, and $O_1 \cap O_2 = \emptyset$. If X is not disconnected, we say that X is connected.

Example 2.3

A line or line segment is connected.

Two open intervals who share a limit point is disconnected.

Two disjoint sets is disconnected.

The set $\left(x, \sin\left(\frac{1}{x}\right)\right)$ is connected.

Theorem 2.4

Let X, Y be metric spaces, and $f : X \to Y$ continuous. If X is connected, then $f(X)$ is connected.

Proof. By contraposition. Assume $\exists O_1, O_2$ disconnected such that $f(X) = O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$.

Take $U_1 = f^{-1}(O_1)$ and $U_2 = f^{-1}(O_2)$. U_1, U_2 are open by continuity.

We show that $U_1 \cap U_2 = \emptyset$. If $x \in U_1 \cap U_2$, then $f(x) \in O_1, f(x) \in O_2$ which is a contradiction.

U_1, U_2 are nonempty since O_1, O_2 are nonempty. Then U_1, U_2 are disconnected. $\square$

Theorem 2.5 (Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ continuous. $\forall L \in \mathbb{R}$ such that $f(a) < L < f(b)$. Then $\exists c \in (a, b)$ such that $f(c) = L$.[2]

Proof. Since $[a, b]$ is connected and f is continuous, then $f([a, b])$ is connected.

Let $O_1 = (-\infty, L) \cap f([a, b])$, $O_2 = (L, \infty) \cap f([a, b])$.

Which are non-empty, open in $f([a, b])$, and disjoint. Then $O_1 \cup O_2 \neq f([a, b])$. So, $L \in f([a, b])$ had better be true. $\square$

[2]This follows the normal high school definition of continuity, 'you can draw it without lifting your pen'.

§2.2 Discontinuities

Definition 2.6 (Discontinuities)

The following definitions:

(1) x_0 is a removable discontinuity if f if $\lim_{x \to x_n} f(x)$ exists but is different from $f(x_0)$.

(2) x_0 is a simple discontinuity if $\lim_{x \to x_0^-} f(x)$, $\lim_{x \to x_0^+} f(x)$ exist but disagree.

(3) x_0 may be a discontinuity of a different type.

Example 2.7

Take

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

§3 March 21, 2023

§3.1 Differentiation

From now on, we fix $f : \mathbb{R} \to \mathbb{R}$, or from an interval to another interval. We'll mostly be dealing with real-valued functions.

Definition 3.1

Let $f : (a, b) \to \mathbb{R}$. We say f is differentiable at $x_0 \in (a, b)$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In that case, we write $f'(x_0)$ for this limit.

Proposition 3.2

Let $f : (a, b) \to \mathbb{R}$ be differentiable at x_0. Then f is continuous at x_0.

Proof.

$$\begin{aligned} \lim_{x \to x_0} f(x) &= f(x_0) + \lim_{x \to x_0} + \lim_{x \to x_0} (f(x) - f(x_0))(x - x_0) \\ &= f(x_0) + \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) \\ &= f(x_0) + f'(x_0) \cdot 0 = f(x_0) \end{aligned}$$

implies $\lim_{x \to x_0} f(x) = f(x_0)$ implies f is continuous at x_0. □

Remark 3.3. The converse of this statement is not true. Continuous does not imply differentiable!

Example 3.4

$f(x) = |x|$ is continuous at at 0 but not differentiable.

Let's focus at $x = 0$.

$$\lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \operatorname{sgn}(x)$$

which doesn't exist.

Proposition 3.5

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions at x. Then the following are differentiable and satisfy:

(1) Sum rule: $(f + g)'(x) = f'(x) + g'(x)$.

(2) Product rule: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

(3) Quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$.

Theorem 3.6 (Chain Rule)

Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to \mathbb{R}$ both be differentiable functions at $x_0, f(x_0)$ respectively. Then $(g \circ f)$ is differentiable at x_0 with derivative

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. We do so in cases.

(i) Suppose exists sequence $(x_n) \in (a, b)$ such that $x_n \to x_0$ if $f(x_n) = f(x_0)$ but $x_n \neq x_0$ for infinitely many n.

Then

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = 0 \end{aligned}$$

and

$$\begin{aligned} (g \circ f)' &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= \lim_{n \to \infty} \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} = 0 \end{aligned}$$

(ii) Every $(x_0) \in (a, b)$ such that $x_n \to x_0$, $x_n \neq x_0$ has $f(x_n) = f(x_0)$ finitely many times.

By continuity, $f(x_n) \to f(x_0)$. Then,

$$\begin{aligned}
(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\
&= \lim_{n \to \infty} \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} \\
&= \lim_{n \to \infty} \left(\frac{g(f(x_n)) - g(f(x_0))}{f(x_n) - f(x_0)} \right) \left(\frac{f(x_n) - f(x_0)}{x_n - x_0} \right) \\
&= \lim_{n \to \infty} \left(\frac{g(f(x_n)) - g(f(x_0))}{f(x_n) - f(x_0)} \right) \cdot \lim_{n \to \infty} \left(\frac{f(x_n) - f(x_0)}{x_n - x_0} \right) \\
&= g'(f(x_0)) \cdot f'(x_0)
\end{aligned}$$

which covers all cases.

$\square$

Example 3.7

Let

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

The derivative at $x = 0$ is

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{f(h)}{h}$$

We try to squeeze this limit,

$$-h = -\frac{h^2}{h} \leq \frac{f(h)}{h} \leq \frac{h^2}{h} = h$$

then

$$\lim_{h \to 0} -h \leq \lim_{h \to 0} \frac{f(h)}{h} \leq \lim_{h \to 0} h$$

which forces $\lim_{h \to 0} \frac{f(h)}{h} = 0$.

This gives $f'(x) = 0$, and

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

so $f'(x)$ is continuous. We say that $f \in \mathcal{C}^1$ (continuous and differentiable). However, the second derivative of f might not exist at $x = 0$.